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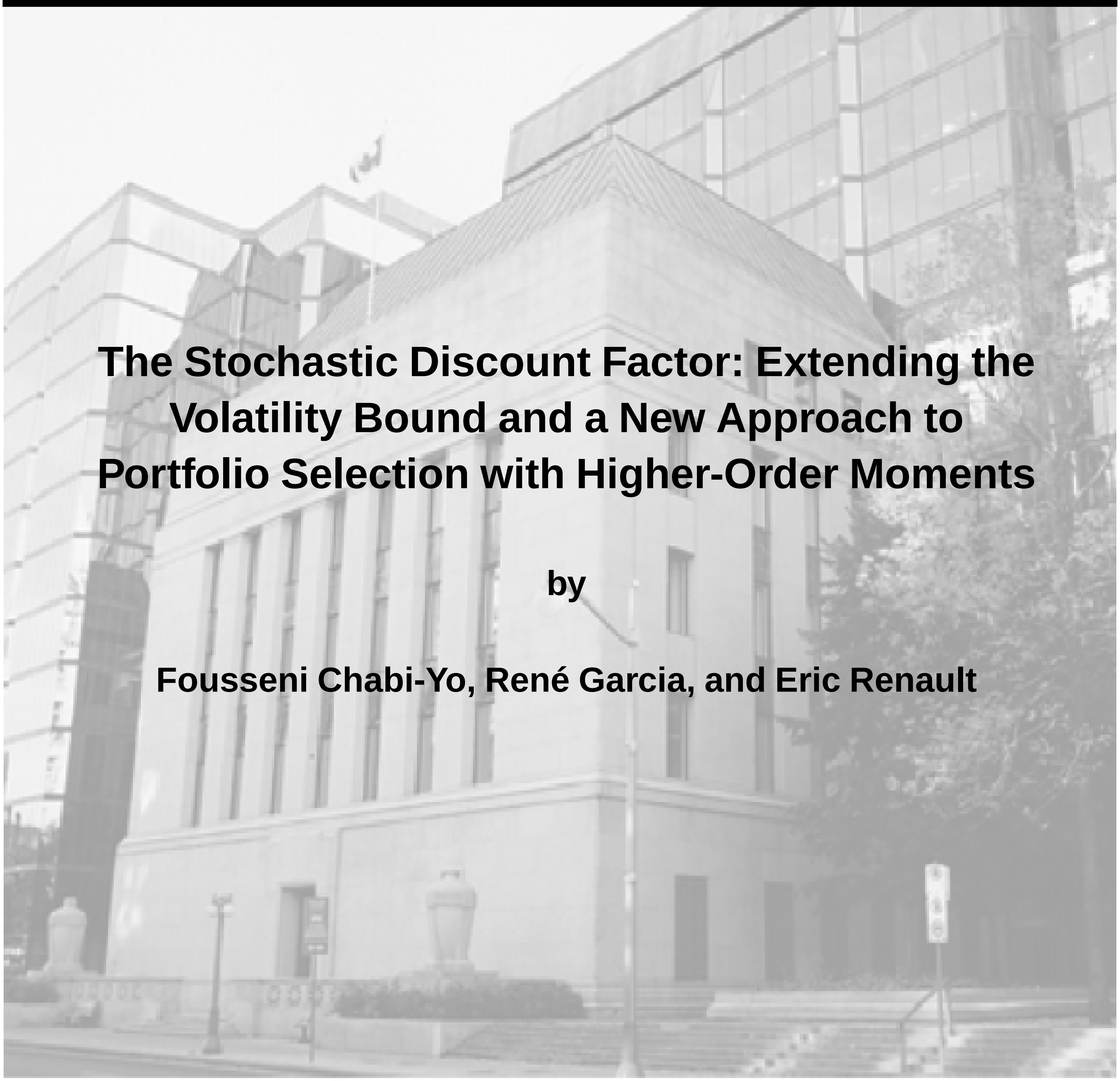

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Fousseni Chabi-Yo,[1] René Garcia,[2] and Eric Renault[3]

[1]Financial Markets Department
Bank of Canada
Ottawa, Ontario, Canada K1A 0G9
fchabiyo@bankofcanada.ca

[2]Département de sciences économiques
Université de Montréal
Montréal, Quebec, Canada H3C 3J7
and CIREQ, CIRANO

[3]University of North Carolina at Chapel Hill
Chapel Hill, NC 27599–3305
and CIREQ, CIRANO

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Abstract

The authors extend the well-known Hansen and Jagannathan (HJ) volatility bound. HJ characterize the lower bound on the volatility of any admissible stochastic discount factor (SDF) that prices correctly a set of primitive asset returns. The authors characterize this lower bound for any admissible SDF that prices correctly both primitive asset returns and quadratic payoffs of the same primitive assets. In particular, they aim at pricing derivatives whose payoffs are defined as non-linear functions of the underlying asset payoffs. The authors construct a new volatility surface frontier in a three-dimensional space by considering not only the expected asset payoffs and variances, but also asset skewness. The intuition behind the authors' portfolio selection is motivated by the duality between the HJ mean-variance frontier and the Markowitz mean-variance portfolio frontier. The authors' approach consists of minimizing the portfolio risk subject not only to portfolio cost and expected return, as usual, but also subject to an additional constraint that depends on the portfolio skewness. In this sense, the authors shed light on portfolio selection when asset returns exhibit skewness.

JEL classification: G11, G12, C61
Bank classification: Financial markets; Market structure and pricing

Résumé

L'objet de l'étude est l'extension du concept bien connu de borne de variance proposé par Hansen et Jagannathan. Alors que ces derniers caractérisent la variance minimale que doit avoir un facteur d'actualisation stochastique admissible pour que soit évalué correctement un ensemble d'actifs primitifs, Chabi-Yo, Garcia et Renault considèrent l'effet qu'a sur cette borne de variance l'ajout de contraintes imposées par l'évaluation correcte des fonctions quadratiques des gains de ces actifs primitifs. Ils abordent ainsi le problème de l'évaluation d'actifs dérivés dont les gains sont par définition des fonctions non linéaires des gains des actifs sous-jacents. Ils trouvent utile de décrire la frontière de variance ainsi obtenue dans un espace à trois dimensions mettant en jeu non seulement les rendements espérés et leur variance, mais aussi leur coefficient d'asymétrie. De même que la frontière de variance de Hansen et Jagannathan présente une relation de dualité avec la frontière efficiente moyenne-variance du choix optimal de portefeuille au sens de Markowitz, la frontière que proposent Chabi-Yo, Garcia et Renault peut être interprétée en termes du choix d'un portefeuille dont le risque est minimisé étant donnés le coût, le rendement espéré et (ce qui est nouveau) le coefficient d'asymétrie du portefeuille. En ce sens, les auteurs donnent un nouvel éclairage au problème de choix de portefeuille en présence de rendements asymétriques.

Classification JEL : G11, G12, C61
Classification de la Banque : Marchés financiers; Structure de marché et fixation des prix

1. Introduction

Hansen and Richard (1987) introduced the concept of the stochastic discount factor (SDF) in the financial econometrics literature and defined a stochastic discount factor as a random variable that discounts payoffs differently in different states of the world. Since their seminal contribution, it has become evident that the empirical implications of asset-pricing models can be characterized through their SDFs (Cochrane 2001). In this context, Hansen and Jagannathan (1991) examine what the data on asset returns may be able to tell us about SDF volatility. They find a lower bound on the volatility of any admissible SDF that prices correctly a set of asset returns. Their bound has been applied to a variety of financial issues. For example, the Hansen and Jagannathan (HJ) volatility bound is used to test whether a particular SDF implied by a model is valid. Barone-Adesi, Gagliardini, and Urga (2004) assume a quadratic specification of the SDF in terms of the market return and test asset-pricing models with co-skewness. They find evidence that asset skewness (co-skewness) is priced in the market through the cost of the squared market return even if the squared market return is not a traded asset. This line of thinking had been initiated by Ingersoll (1987) and pursued by Harvey and Siddique (2000) and Dittmar (2002). They examine extensions of the capital-asset-pricing model (CAPM) framework by considering asset skewness. Ingersoll (1987), assuming higher skewness is preferred, shows that a decrease in co-skewness requires an increase in expected return to induce the same holding of the asset at the margin. Furthermore, if we use a Taylor series of derivatives' payoff functions as quadratic functions of the underlying asset return, the price of the derivatives is a function of the cost of the squared return and this cost is closely related to return skewness. The cost of the squared portfolio return is, therefore, particularly relevant when pricing derivatives. Since the HJ volatility bound considers admissible SDFs that price correctly only a set of asset returns, it appears useful to construct a new variance bound for any admissible SDF that prices correctly not only a set of primitive assets, but their squared returns.

The first contribution of this paper is to find such a lower bound. While HJ minimize the SDF variance for a given SDF mean under the assumption that the admissible SDFs price correctly a set of primitive asset returns, we minimize the SDF variance for a given SDF mean under the assumption that the admissible SDFs price correctly not only a set of primitive asset returns but also their squared returns. Our variance bound tightens the HJ volatility bound by an additional quantity that is a function of the assets' co-skewness and the cost of the squared primitive asset returns. We derive necessary and sufficient conditions to get the well-known HJ volatility bound as a particular case. In this more general setting, our minimum-variance SDF can be rewritten as a quadratic function of asset returns. By this, we mean a linear combination of two vectors: R and $R^{(2)}$, where R is a vector of primitive asset returns and $R^{(2)}$ is a vector of the squared primitive asset returns whose components are of the form R_iR_j with $i \leq j$. When R is the market return, we get a quadratic specification of the SDF in terms of the market return, which is often used to underline the importance of skewness (co-skewness) in asset-pricing models (Ingersoll 1987, Harvey and Siddique 2000, and Dittmar 2002). We use the return on the Standard & Poor's 500 stock index and the commercial paper index from 1889 to 1994 to illustrate our SDF volatility surface frontier. We also use the consumption on non-durables and services over the same period to relate the constant relative risk aversion (CRRA) and Epstein and Zin (1989) preference models to our volatility bound for particular values of the relative risk-aversion coefficient. We illustrate how our SDF variance frontier tightens the HJ variance frontier and makes the equity-premium puzzle even more difficult to solve.

The second contribution of this paper is to offer a new approach to portfolio selection with higher-order moments. This approach is based on factors that span our minimum-variance SDF. The intuition behind our portfolio selection is motivated by the duality between the HJ minimum-variance SDF and Markowitz mean-variance analysis (Campbell, Lo, and MacKinlay 1997). Since we have found a minimum-variance SDF that tightens the HJ minimum-variance SDF, it is of interest to also base our portfolio selection on our minimum-variance SDF. Our approach consists of minimizing the portfolio risk subject to the portfolio expected return and an additional constraint (the cost of the squared portfolio return) that depends on the portfolio skewness. We seek to determine the conditions under which our portfolio selection is observationally equivalent to the standard portfolio selection under skewness. We then generalize the standard approach to portfolio selection under skewness, which consists of minimizing the portfolio risk subject to the portfolio expected return and skewness (see Lai 1991; de Athayde and Flores 2004). Our more general approach is relevant, first because it provides a formal bridge between the SDF variance bound and portfolio selection under higher-order moments. Second, it shows that the standard approach of portfolio selection under skewness may overlook an important factor.

We also provide an empirical illustration of portfolio selection with higher-order moments. We use daily asset returns for four individual firms. Our approach to portfolio selection depends on the cost of the squared asset returns. To compute this cost, we assume that the joint process of the SDF and asset returns is lognormally distributed. The lognormal distribution is more flexible and allows for skewness in asset returns. For example, many asset-pricing tests assume that the joint process of SDF-asset returns is conditionally jointly lognormal. Moreover, diffusion models imply a locally lognormal distribution. Our results suggest that the cost of the squared portfolio return and portfolio higher-order moments have a significant impact on the portfolio mean-variance frontier.

The rest of this paper is organized as follows. Section 2 provides the theoretical background and an empirical illustration of the generalized SDF variance bound. Section 3 describes an approach to portfolio selection based on factors that span our minimum-variance SDF. Section 4 provides an empirical illustration of portfolio selection with higher-order moments. Section 5 offers some conclusions.

2. The Minimum-Variance Stochastic Discount Factor

In this section, we first review the HJ volatility bound and derive the SDF variance bound under higher-order moments. In section 2.2, we describe conditions under which the cost of the squared returns affects the variance bound and note some empirical implications of our new bound. Section 2.3 discusses the variance bound when we restrict admissible SDFs to be positive.

2.1 The general framework

In this subsection, we construct a new bound on the volatility of any admissible SDF that tightens the HJ volatility bound. By SDF, we mean a random variable that can be used to compute the market price of an asset today by discounting payoffs differently in different states of the world in the future. HJ propose a way to find the lower bound on the volatility of any SDF that prices correctly a set of primitive asset returns. Their approach treats the unconditional mean of the SDF as an unknown parameter, $\overline{m}$. For each possible parameter $\overline{m}$, HJ form a candidate SDF, $m_{HJ}\left(\overline{m}\right)$, as a linear combination of asset returns, and they show that the variance of $m_{HJ}\left(\overline{m}\right)$ represents a lower bound on the variance of any SDF that has mean $\overline{m}$ and satisfies:

$$EmR = l,$$

where l represents an N-vector column of 1 and R is a set of N primitive asset returns. Let $\mathcal{F}_1\left(\overline{m}\right)$ denote the set of SDFs that have mean $\overline{m}$ and that price correctly R. Therefore,

$$\mathcal{F}_1\left(\overline{m}\right) = \left\{m \in L^2 : Em = \overline{m},\ EmR = l\right\}.$$

Thus, $m_{HJ}\left(\overline{m}\right)$ is the solution to the problem:

$$\underset{m\in\mathcal{F}_1(\overline{m})}{Min}\ \sigma\left(m\right).$$

HJ show that

$$m_{HJ}\left(\overline{m}\right) = \overline{m} + \left(l - \overline{m}ER\right)' \Omega^{-1}\left(R - ER\right),$$

and

$$Var\left[m_{HJ}\left(\overline{m}\right)\right] = \left(l - \overline{m}ER\right)' \Omega^{-1}\left(l - \overline{m}ER\right),$$

where Ω is the covariance matrix of the asset returns. The N assets are risky and no linear combination of the returns in R is equal to one with probability one, so that Ω is non-singular. Using the HJ volatility bound, it is possible to derive an admissible region for mean and standard deviations of candidate SDFs using only data on asset returns. By plotting these regions, the HJ approach provides an appealing graphical technique by which to gauge the specification of many asset-pricing models. It appears important, however, for any admissible SDF to price correctly not only a set of primitive assets, but also payoffs that are non-linear functions of primitive assets' payoffs. For instance, a Taylor series expansion of derivatives' payoffs around a benchmark return will imply, in general, that the cost of squared portfolio returns is relevant when pricing derivatives.

$r_p = \omega' R$ represents a portfolio return, where $\omega = (\omega_1, \omega_2, ...\omega_N)'$ is a vector of portfolio weights that satisfies $\omega' l = 1$, with $l = (1, 1, ...1)'$. The squared return of the portfolio can be represented by:

$$r_p^2 = \left(\omega' R\right)^2 = (\omega \otimes \omega)' (R \otimes R),$$

where $\otimes$ stands for the Kronecker product. The cost of the squared portfolio return is, therefore,

$$\begin{aligned} C\left(r_p^2\right) &= Emr_p^2 \\ &= (\omega \otimes \omega)' Em (R \otimes R) \\ &= \omega^{(2)'} EmR^{(2)}, \end{aligned}$$

where $\omega^{(2)}$ represents a column vector whose components are of the form,

$$\omega_{ij} = \begin{cases} 2\omega_i\omega_j \text{ if } i < j \\ \omega_i^2 \text{ if } i = j \end{cases},$$

and $R^{(2)}$ represents a column vector, the components of which are of the form R_iR_j with $i \leq j$. It can be observed that the cost of the squared portfolio return is a function of the cost of the "squared" asset returns, $R^{(2)}$.[1] The question we wish to resolve is whether we can tighten the HJ volatility bound by considering any admissible SDF that correctly prices payoffs that can be expressed as a quadratic function of the primitive assets. The idea is to consider a set of SDFs that correctly price the N asset returns, R, and the "squared" asset returns, $R^{(2)}$. To see why it is interesting to consider SDFs that correctly price these non-linear payoffs, consider the payoff $g(r_p)$ and assume that it can be approximated by its fitted linear regression on r_p and r_p^2:

$$g(r_p) = Eg(r_p) + a(r_p - Er_p) + b\left(r_p^2 - \frac{cov\left(r_p, r_p^2\right)}{Var(r_p)}(r_p - Er_p)\right).$$

The price of this payoff is:

$$\pi_p = Emg(r_p) = \overline{m}Eg(r_p) + a(1 - \overline{m}Er_p) + b\left(C\left(r_p^2\right) - \frac{cov\left(r_p, r_p^2\right)}{Var(r_p)}(1 - \overline{m}Er_p)\right).$$

This last expression shows that the price of the squared portfolio return, $C\left(r_p^2\right)$, is relevant in computing the price of the payoff, $g(r_p)$. If $\mathcal{F}_2(\overline{m}, \eta)$ denotes a set of SDFs that correctly price R and $R^{(2)}$, we have,

$$\mathcal{F}_2(\overline{m}, \eta) = \left\{m \in L^2 : Em = \overline{m},\ EmR = l,\ EmR^{(2)} = \eta\right\},$$

[1] For portfolio algebra, using the inverse of covariance matrices, we prefer to use $R^{(2)}$ rather than $R \otimes R$, since the latter has a singular covariance matrix.

where η denotes the vector of prices of squared returns. Note that $\mathcal{F}_2(\overline{m},\eta) \subset \mathcal{F}_1(\overline{m})$. Intuitively, we exclude any admissible SDF that does not correctly price derivatives with payoffs that can be written as a quadratic function of a set of primitive assets. We then treat the unconditional mean, $\overline{m}$, of the SDF and the cost, η, of the "squared" primitive asset, $R^{(2)}$, as unknown parameters. For each $\overline{m}$ and η, we form a candidate SDF, $m^{mvs}(\eta,\overline{m})$, as a quadratic function of asset returns:

$$m^{mvs}(\eta,\overline{m}) = \alpha(\eta,\overline{m}) + \beta(\eta,\overline{m})' R + \gamma(\eta,\overline{m})' R^{(2)}, \tag{2.1}$$

with

$$\alpha(\eta,\overline{m}) = \overline{m} - \beta(\eta,\overline{m})' ER - \gamma(\eta,\overline{m})' ER^{(2)},$$

since $Em^{mvs}(\eta,\overline{m}) = \overline{m}$. Therefore, we exploit the pricing formulas $E(Rm) = l$ and $E\left(R^{(2)}m\right) = \eta$ to compute the parameters,

$$\begin{aligned}
\beta(\eta,\overline{m}) &= \Omega^{-1}(l - \overline{m}ER) - \Omega^{-1}\Lambda\gamma(\eta,\overline{m}), \\
\gamma(\eta,\overline{m}) &= \left[\Sigma - \Lambda'\Omega^{-1}\Lambda\right]^{-1}\left[\eta - \overline{m}ER^{(2)} - \Lambda'\Omega^{-1}(l - \overline{m}ER)\right],
\end{aligned}$$

with

$$\begin{aligned}
\Sigma &= ER^{(2)}\left(R^{(2)} - ER^{(2)}\right)', \\
\Lambda' &= E\left(R^{(2)} - ER^{(2)}\right)R'.
\end{aligned}$$

Note that Λ is related to the notion of co-skewness (see Ingersoll 1987; Harvey and Siddique 2000). The expansion $\Psi = \Sigma - \Lambda'\Omega^{-1}\Lambda$ denotes the residual covariance matrix in the regression of $R^{(2)}$ on R. We assume that the matrix Ψ is non-singular; that is, no squared returns are redundant with respect to the primitive assets. This assumption will be maintained hereafter for the sake of notational simplicity. A simple application of the HJ argument to the vector $\left[R, (diag\ \eta)^{-1} R^{(2)}\right]$ of returns (where $diag\ \eta$ denotes the diagonal matrix with coefficients defined by the components of η) ensures that $m^{mvs}(\eta,\overline{m})$ gives the lower bound on the volatility in $\mathcal{F}_2(\overline{m},\eta)$. That is, it solves:

$$\min_{m \in \mathcal{F}_2(\overline{m},\eta)} \sigma(m).$$

To compare this minimum-variance SDF with the HJ minimum-variance SDF associated with only the vector R of returns, we rewrite $m^{mvs}(\eta,\overline{m})$ as a function of the HJ minimum-variance SDF.

Proposition 2.1 *The minimum-variance SDF among any admissible SDFs that correctly price not only a set of primitive assets but also derivatives, the payoffs of which can be written as a quadratic function of the same primitive assets, can be expressed as follows:*

$$m^{mvs}(\eta,\overline{m}) = m_{HJ}(\overline{m}) + \gamma(\eta,\overline{m})'\left[R^{(2)} - ER^{(2)} - \Lambda'\Omega^{-1}(R - ER)\right],$$

where

$$\gamma(\eta,\overline{m}) = \left[\Sigma - \Lambda^{'}\Omega^{-1}\Lambda\right]^{-1}\left[\eta - \overline{m}ER^{(2)} - \Lambda^{'}\Omega^{-1}(l - \overline{m}ER)\right].$$

PROOF. The proof is similar to the proof of the HJ minimum-variance SDF, $m_{HJ}(\overline{m})$; see Hansen and Jagannathan (1991). ■

We next discuss the necessary and sufficient conditions to get the HJ minimum-variance SDF.

Proposition 2.2 *The minimum-variance SDF,* $m^{mvs}(\eta,\overline{m})$, *collapses to the HJ minimum-variance SDF,* $m_{HJ}(\overline{m})$, *if and only if*

$$\eta = \overline{m}ER^{(2)} + \Lambda^{'}\Omega^{-1}(l - \overline{m}ER).$$

PROOF. Of course, if $\gamma(\eta,\overline{m}) = 0$, we have $m^{mvs}(\eta,\overline{m}) = m_{HJ}(\overline{m})$. Conversely, assume that $m^{mvs}(\eta,\overline{m}) = m_{HJ}(\overline{m})$; thus, it follows that

$$\gamma(\eta,\overline{m})^{'}\left[R^{(2)} - ER^{(2)} - \Lambda^{'}\Omega^{-1}(R - ER)\right] = 0.$$

If we premultiply this equality by $m^{mvs}(\eta,\overline{m})$, we get

$$\gamma(\eta,\overline{m})^{'} m^{mvs}(\eta,\overline{m})\left[R^{(2)} - ER^{(2)} - \Lambda^{'}\Omega^{-1}(R - ER)\right] = 0.$$

Taking the expectation of this quantity, it is easy to show that

$$\eta - \overline{m}ER^{(2)} - \Lambda^{'}\Omega^{-1}(l - \overline{m}ER) = 0.$$

This implies that $\gamma(\eta,\overline{m}) = 0$. ■

Note that propositions 2.1 and 2.2 have been derived under the maintained assumption that squared returns are not redundant assets. That is, $R^{(2)}$ does not coincide with its affine regression, R:

$$R^{(2)} - ER^{(2)} - \Lambda^{'}\Omega^{-1}(R - ER).$$

However, even when this residual has zero price (i.e., its product by the SDF has a zero expectation), we see from proposition 2.2 that $m^{mvs}(\eta,\overline{m})$ and $m_{HJ}(\overline{m})$ coincide.

Our volatility bound can be used to assess the specification of a particular asset-pricing model, as is usually done with the HJ volatility bound. But our bound is tighter:

$$\sigma\left[m^{mvs}(\eta,\overline{m})\right] \geq \sigma\left[m_{HJ}(\overline{m})\right] \text{ for all } \eta. \tag{2.2}$$

To see how our volatility bound can be used to check whether a particular asset-pricing model explains asset returns, let us consider a proposed SDF, $m(x)$, where x represents a set of relevant variables; for example, the ratio of consumption, $x = \frac{C_{t+1}}{C_t}$, or the first difference of consumption, $C_{t+1} - C_t$. To gauge whether the proposed SDF passes our volatility bound, we need to first compute $\eta = Em(x)R^{(2)}$ and $Em(x)$, and then check whether $\sigma[m(x)] \geq \sigma[m^{mvs}(\eta, Em(x))]$. If the proposed SDF passes the HJ volatility bound but not our variance bound, the proposed SDF variance is too low and this SDF cannot correctly price derivatives, the payoffs of which are a quadratic function of the primitive assets. Since the price of such derivatives can be written as a function of the cost of $R^{(2)}$ and this cost is a function of asset skewness, the failure of the proposed SDF is akin to a failure to price skewness correctly.

2.2 Why does the cost of squared returns matter?

By the inequality (2.2), we realize that our variance bound is greater than the HJ volatility bound. The first question we ask is: are there pricing conditions under which our variance bound coincides with the HJ volatility bound? Under these conditions, the cost of squared returns would not matter and we would have failed to shed more light on the SDF variance bound. Proposition 2.3 summarizes this issue.

Proposition 2.3 *Consider the linear regression of the squared returns, $R^{(2)}$, on the return, R; that is:*

$$EL\left[R^{(2)}|R\right] = ER^{(2)} + \Lambda^{'}\Omega^{-1}(R - ER),$$

and

$$\eta^{*} = \overline{m}ER^{(2)} + \Lambda^{'}\Omega^{-1}(l - \overline{m}ER),$$

the price of this regression. Then, there exists $\eta > 0$ such that

$$\sigma[m^{mvs}(\eta, \overline{m})] = \sigma[m_{HJ}(\overline{m})],$$

if and only if $\eta^{*} > 0$.

Proof. We have $\gamma(\eta, \overline{m}) = \left[\Sigma - \Lambda^{'}\Omega^{-1}\Lambda\right]^{-1}[\eta - \eta^{*}]$. Then, if $\eta^{*} > 0$, $\eta = \eta^{*}$ implies that $\gamma(\eta, \overline{m}) = 0$. We therefore have $\sigma[m^{mvs}(\eta^{*}, \overline{m})] = \sigma[m_{HJ}(\overline{m})]$. Conversely, assume that there exists $\eta > 0$ such that $\sigma[m^{mvs}(\eta, \overline{m})] = \sigma[m_{HJ}(\overline{m})]$. This implies that $\gamma^{'}(\eta, \overline{m})\left[\Sigma - \Lambda^{'}\Omega^{-1}\Lambda\right]\gamma(\eta, \overline{m}) = 0$. But,

$$\gamma^{'}(\eta, \overline{m})\left[\Sigma - \Lambda^{'}\Omega^{-1}\Lambda\right]\gamma(\eta, \overline{m}) = (\eta - \eta^{*})^{'}\left[\Sigma - \Lambda^{'}\Omega^{-1}\Lambda\right](\eta - \eta^{*}).$$

Therefore, $(\eta - \eta^{*})^{'}\left[\Sigma - \Lambda^{'}\Omega^{-1}\Lambda\right](\eta - \eta^{*}) = 0$. Since we assume in this paper that the matrix $\Sigma - \Lambda^{'}\Omega^{-1}\Lambda$ is non-singular, we conclude by the Cauchy-Schwarz inequality that $\Sigma - \Lambda^{'}\Omega^{-1}\Lambda$ is positive definite; therefore, $\eta^{*} = \eta > 0$. ■

In other words, when $\eta^{*} < 0$, $\sigma[m^{mvs}(\eta, \overline{m})] > \sigma[m_{HJ}(\overline{m})]$ for all $\eta > 0$. Then, taking into account the cost of squared returns will always have a significant impact on the volatility bound.

Kan and Zhou (2003) propose an alternative way to tighten the HJ volatility bound. They assume that they can find a vector, x, of state variables such that the conditional expectation of $m_{HJ}(\overline{m})$, given x, coincides with its affine regression. Under this maintained assumption, Kan and Zhou show that any admissible SDF $m(x)$ that is a deterministic function of x has a larger volatility than a bound $\sigma^2[m_{KZ}]$ defined by:

$$\sigma^2[m(x)] \geq \sigma^2[m_{KZ}] = \frac{1}{\rho^2_{m_{HJ},x}}\sigma^2[m_{HJ}(\overline{m})],$$

where $\rho_{m_{HJ},x}$ is the multiple linear correlation coefficient between $m_{HJ}(\overline{m})$ and x. By considering $x = \left[R, R^{(2)}\right]$, we can then claim that:

$$\inf_{\eta} \sigma^2[m^{mvs}(\eta, \overline{m})] \geq \sigma^2[m_{KZ}].$$

Therefore, Kan and Zhou's volatility bound does not make our bound irrelevant. The cost of squared returns may matter significantly.

Empirical illustrations show that the cost of squared returns may be important. We first consider the annual excess simple return of the Standard & Poor's 500 stock index over the commercial paper rate from 1889 to 1994. In this case, $q = 1$ and our SDF variance bound is easy to illustrate graphically. Figures 1 and 2 illustrate our variance bound surface and the HJ volatility bound, respectively. These figures show that the cost of the squared asset excess return has a significant impact on the SDF mean-standard deviation frontier. For example, for an SDF mean in the neighbourhood of 1, the standard deviation of the HJ minimum-variance SDF is about 0.3, whereas the standard deviation of our minimum-variance SDF is greater than 0.6 for any positive value of the squared return cost. According to proposition 2.3, this should be a case where the cost, η^*, of the affine regression of $R^{(2)}$ on R is negative. Furthermore, when the SDF mean is in the neighbourhood of 1, our minimum-variance SDF standard deviation depends highly on the cost of the squared asset excess return. Thus, the cost of squared returns is relevant for determining the SDF variance bound. Similarly to the HJ volatility bound, our volatility bound can be used to illustrate whether a particular asset-pricing model fails to explain a set of asset returns. To provide this illustration, we consider several consumption-based models. The first model assumes that there is a representative agent who maximizes a time-separable power utility function, so that:

$$u(C_{t+1}) = \frac{C_{t+1}^{1-\alpha} - 1}{1-\alpha},$$

where α is the coefficient of relative risk aversion and C_{t+1} is aggregate consumption. Therefore, it can be shown that the representative agent's optimization problem yields an SDF of the form:

$$m_{t+1} = \beta\frac{u'(C_{t+1})}{u'(C_t)},$$

where $\beta \in (0,1)$ is a subjective discount parameter. For this CRRA preference model, we set $\beta = 0.95$. Using consumption on non-durables and services over the same period, 1889 to 1994, Campbell, Lo, and MacKinlay (1997) show that the variance of m_{t+1} falls in the HJ feasible region if the relative risk-aversion coefficient, α, is greater than 25. This is shown in Figure 3. When we vary α exogenously from 0 to 27, the point $(Em_{t+1}, \sigma(m_{t+1}))$ does not fall into the feasible region until the coefficient of the relative risk aversion reaches a value of 25.

Since our SDF variance bound is greater than the HJ variance bound, it is clear that, for $\alpha \leq 24$, the point $(Em_{t+1}, \sigma(m_{t+1}), \eta)$ with $\eta = Em_{t+1}R^{(2)}$ does not enter into our feasible region. We need to check whether any particular relative risk aversion, $\alpha \geq 25$, produces a point $(Em_{t+1}, \sigma(m_{t+1}), \eta)$ that enters our feasible region. To proceed with our graphical illustration, for $\alpha = 25$ and $\alpha = 27$, we compute η and find the corresponding feasible region. We check, thereafter, whether the point $(Em_{t+1}, \sigma(m_{t+1}))$ enters our feasible region. While Figure 4 shows that, for various relative risk-aversion coefficients, our variance bound never coincides with the HJ volatility bound, the two bounds nevertheless provide the same conclusion about the candidate SDFs produced by this model.

We repeat the same calibration exercise using Epstein and Zin's (1989) state non-separable preferences. Following Epstein and Zin (1989), we assume that the state non-separable preferences are given by $V_t = U[C_t, E_tV_{t+1}]$, where

$$U[C_t, V] = \frac{\left[(1-\beta)C_t^{1-\rho} + \beta\left[1 + (1-\beta)(1-\alpha)V\right]^{\frac{1-\alpha}{1-\rho}}\right] - 1}{(1-\beta)(1-\alpha)}.$$

The elasticity of intertemporal substitution is $1/\rho$. The representative agent SDF is

$$m_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\frac{1-\alpha}{1-\rho}} R_{mt+1}^{\left(\frac{1-\alpha}{1-\rho}\right)-1},$$

where R_{mt+1} is the return on the market portfolio. Figure 5 plots the bound and the representative agent SDF volatilities for Epstein and Zin's (1989) consumption-based model. For this consumption-based model, the parameters used are $\beta = 0.96$. We use the same data set as in the CRRA case. Figure 5 shows that, for $\beta = 0.96$, $(\rho, \alpha) = (3.05, 6.86)$, the point $(Em_{t+1}, \sigma(m_{t+1}))$ enters the HJ feasible region, but it does not enter our feasible region. This means that taking into account the cost of quadratic derivatives makes the equity-premium puzzle even more difficult to solve. For reasonable values of the preference parameters, Figure 5 also shows that our variance bound never coincides with the HJ volatility bound. This underlines why the cost of squared returns should be taken into account in asset-pricing models.

Next, we consider a model with state dependence in preferences. Several authors (e.g., Gordon and St-Amour 2000; Melino and Yang 2003) point to counter-cyclical risk aversion as a potential source of misspecification that may account for the equity-premium puzzle. It is of interest to check whether these models can explain this puzzle when using our variance bound on admissible SDF. We consider Gordon and St-Amour's (2000) and Melino and Yang's (2003) state-dependent preference model.

Gordon and St-Amour's (2000) SDF is of the form

$$m_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\alpha(U_t)}\left(\frac{C_{t+1}}{\theta}\right)^{\alpha(U_t)-\alpha(U_{t+1})},$$

where the coefficient of relative risk aversion depends on a latent state variable, U_t, and $\frac{C_{t+1}}{\theta}$ is the ratio of the next period's level of consumption to a scale parameter, θ. For the state variable, we set the transition matrix to[2]

$$\Pi = \begin{bmatrix} 0.9909 & 0.0061 \\ 0.0091 & 0.9939 \end{bmatrix}.$$

Since the frontiers are very close under the two bounds, we find that, when the implied Gordon and St-Amour SDF passes the HJ volatility bound, it also passes our variance bound. We report (see Figure 6) only the case $\alpha = (3.7, 2.23)$, $\theta = 12,\ 18$. Melino and Yang (2003) generalize the model of Epstein and Zin (1989) by allowing the representative agent to display state-dependent preferences, and show that these preferences can add to the explanation of the equity-premium puzzle. They consider several state-dependent preference cases: state-dependent CRRA, state-dependent elasticity of intertemporal substitution (EIS), and state-dependent subjective discount parameter β. Without loss of generality, we consider Melino and Yang's (2003) SDF when the EIS and the subjective discount parameter β are constant:

$$m_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\frac{1-\alpha(U_t)}{1-\rho}} R_{mt+1}^{\left(\frac{1-\alpha(U_t)}{1-\rho}\right)-1}.$$

Figure 7 shows two examples of SDF bounds. For fixed values of $(\beta, \rho) = (0.98, 3.58)$, the first example shows that, for the state-dependent preferences parameter $\alpha = (7.8, 9.4)$, the state-dependent implied SDF passes the HJ volatility bound but does not pass our bound, whereas in the second example, $\alpha = (8.8, 9.85)$ produces an SDF that passes both bounds. In the next subsection, we provide insight into the SDF variance bound under a positivity constraint.

2.3 Positivity constraint on the SDF

So far, we have ignored the arbitrage restriction that an admissible SDF must be non-negative. HJ show that, when an unconditionally riskless asset exists, it is straightforward to find the HJ minimum-variance SDF with a non-negativity constraint. But they show that this SDF may not be unique. In our case, when the unconditionally riskless asset exists, it can be shown that the minimum-variance SDF with positivity constraint is:

$$m^{mvs}(\eta)^{+} = \left(\widetilde{\beta}(\eta)' R + \widetilde{\gamma}(\eta)' R^{(2)}\right)^{+},$$

where $x^{+} = \max(0, x)$ represents the non-negative part of x. The parameters $\widetilde{\beta}(\eta)$ and $\widetilde{\gamma}(\eta)$ can be computed by solving the non-linear equations:

$$\begin{aligned} ERm^{mvs}(\eta)^{+} &= l, \\ ER^{(2)}m^{mvs}(\eta)^{+} &= \eta. \end{aligned}$$

[2] In this matrix, the probability of staying in state 1 is 0.9909 and the probability of staying in state 2 is 0.9939.

These two equations are non-linear in the parameter vectors $\widetilde{\beta}(\eta)$ and $\widetilde{\gamma}(\eta)$, and the solution $(\widetilde{\beta}(\eta), \widetilde{\gamma}(\eta))$ cannot be represented in terms of matrix manipulations. Similarly to HJ, it can be shown that this solution exists but may not be unique. Once this solution is found, however, it is easy to show that $m^{mvs}(\eta)^{+}$ has a minimum variance among any admissible SDF in $\mathcal{F}_2^+(\eta)$, where

$$\mathcal{F}_2^+(\eta) = \left\{ m \in L^2 : m > 0,\ EmR = l,\ EmR^{(2)} = \eta \right\}.$$

To understand this more clearly, consider any other non-negative admissible SDF in $\mathcal{F}_2^+(\eta)$ and note that

$$\begin{aligned} E\left[m^{mvs}(\eta)^{+} m\right] &= E\left[m\left(\widetilde{\beta}(\eta)' R + \widetilde{\gamma}(\eta)' R^{(2)}\right)^{+}\right] \\ &\geq \widetilde{\beta}(\eta)' EmR + \widetilde{\gamma}(\eta)' EmR^{(2)} \\ &= \widetilde{\beta}(\eta)' Em^{mvs}(\eta)^{+} R + \widetilde{\gamma}(\eta)' Em^{mvs}(\eta)^{+} R^{(2)} \\ &= E\left[\left(m^{mvs}(\eta)^{+}\right)^2\right]. \end{aligned}$$

It follows that

$$Em^2 \geq E\left[\left(m^{mvs}(\eta)^{+}\right)^2\right],$$

and

$$\sigma(m) \geq \sigma\left(m^{mvs}(\eta)^{+}\right).$$

HJ find a similar inequality, but in their framework,

$$\sigma(m) \geq \sigma\left(m_{HJ}^{+}\right),$$

for any admissible SDF in $\mathcal{F}_1^+ = \left\{m \in L^2 : m > 0, EmR = l\right\}$, where $m_{HJ}^{+} = \left(\beta_{HJ}' R\right)^{+}$. Since $\mathcal{F}_2^+(\eta) \subset \mathcal{F}_1^+$, it is straightforward to show that:

$$\sigma\left(m^{mvs}(\eta)^{+}\right) \geq \sigma\left(m_{HJ}^{+}\right).$$

Therefore, when the riskless asset exists, and if we use a non-negativity constraint on m, our variance bound also tightens the HJ variance bound. Following the same idea as in Hansen and Jagannathan (1991), this result can be generalized to deal with the case in which there is no unconditionally riskless asset. In the rest of this paper, we work without a positivity constraint on admissible SDFs.

Motivated by the duality between the HJ frontier and the Markowitz mean-variance portfolio frontier, we offer, in the next section, an approach to portfolio selection based on our minimum-variance SDF surface frontier.

3. Portfolio Selection

Markowitz mean-variance analysis is the central tenet of portfolio selection in financial theory. Since any asset-pricing model can be represented by an SDF model, a number of papers establish a connection between

Markowitz mean-variance analysis and the HJ bound on the SDF volatility (e.g., Campbell, Lo, and MacKinlay 1997; Nijman and de Roon 2001; Penaranda and Sentana 2001). The leading assumption in Markowitz mean-variance analysis is that investors are interested in three characteristics of their portfolio: expected payoff, cost, and variance. Under these assumptions, it can be shown that the HJ minimum-variance SDF is spanned by two factors and that the Markowitz optimization problem (which entails minimizing the (unit cost) portfolio variance subject to the portfolio expected return) yields an optimal mean-variance portfolio that can be written as a function of the same two factors.

In this section, we assume that investors are interested not only in these three characteristics of their portfolio, but also in the cost of their squared portfolio return.

We first use these four characteristics to decompose the SDF as a function of factors that we use to provide an approach to portfolio selection. Our main contribution is to show that the recent approach to portfolio selection based on mean-variance-skewness may miss an important factor.

3.1 An SDF decomposition

Let $\mathcal{P}_N$ be the set of payoffs that is given by the linear span of primitive assets, and let $\mathcal{G}_N$ be the set of the payoffs that is given by the linear span of "squared" primitive assets, $R^{(2)}$. The elements of $\mathcal{P}_N$ will be of the form

$$\sum_{i=1}^{N} \omega_i R_i.$$

Similarly, the elements of $\mathcal{G}_N$ will be of the form

$$\sum_{i\leq j}^{N} \omega_{ij} R_i R_j.$$

$\mathcal{P}_N$, $\mathcal{G}_N$ are closed linear subspaces of L^2, where L^2 denotes the Hilbert space under the mean-square inner product defined as $\langle x, y\rangle = Exy$ and the associated mean-square norm $\langle x, x\rangle^{1/2}$ with $x,\ y \in L^2$. Assume that investors are interested in at least four characteristics of their portfolio $p = \omega^{'} R$: the (normalized) cost of their portfolio, their portfolio expected payoff value, the variance of their portfolio payoff, and the cost of their squared portfolio returns, which are given by $C(p) = \omega^{'} l$, $E(p) = \omega^{'} \nu$, $V(p) = \omega^{'} \Omega \omega$, and $\widetilde{C}\left(p^2\right)$, respectively.

For convenience, we denote

$$\begin{aligned} \Gamma &= ERR^{'}, \\ \Gamma^{(2)} &= ER^{(2)} R^{(2)^{'}}. \end{aligned}$$

Under the law of one price, we can interpret both $C(.)$, $E(.)$ as linear functionals that map the elements of $\mathcal{P}_N$ into the real line. In this sense, the Riesz representation theorem says that there exist two unique elements of $\mathcal{P}_N$ such that:

$$C(p) = E\left(p^{+}p\right) \quad \forall p \in \mathcal{P}_N,$$

with

$$p^{+} = a^{+'}R, \text{ with } a^{+'} = l^{'}\Gamma^{-1}, \tag{3.3}$$

and

$$E(p) = E\left(p^{++}p\right) \quad \forall p \in \mathcal{P}_N, \tag{3.4}$$

with

$$p^{++} = a^{++'}R, \text{ with } a^{++'} = \nu^{'}\Gamma^{-1}.$$

Similarly, $\widetilde{C}(.)$ can be viewed as a linear functional that maps the elements of $\mathcal{G}_N$ into the real line. The Riesz representation theorem again implies that there exists a unique element of $\mathcal{G}_N$ such that:

$$\widetilde{C}(\widetilde{p}) = E(p^{*}p) \quad \forall p \in \mathcal{G}_N, \tag{3.5}$$

with

$$p^{*} = a^{*'}R^{(2)},$$

where $a^{*'} = \eta^{'}\left[\Gamma^{(2)}\right]^{-1}$. The following theorem shows that these three vectors p^{+}, p^{++}, and p^{*} are able to span the minimum-variance SDFs.

Theorem 3.1 *For any $\eta \neq \eta^{*}$, the minimum-variance SDF $m^{mvs}(\eta, \overline{m})$ can be decomposed as:*

$$m^{mvs}(\eta, \overline{m}) = m_{HJ}(\overline{m}) + cF_3,$$

with $m_{HJ}(\overline{m}) = \overline{m} + aF_1 + bF_2$, where

$$\begin{aligned} F_1 &= p^{+} - Ep^{+}, \\ F_2 &= p^{++} - EL\left[p^{++}|F_1\right], \\ F_3 &= p^{*} - EL\left[p^{*}|F_1, F_2\right], \end{aligned}$$

and

$$\begin{aligned} a &= \frac{l^{'}\Gamma^{-1}l - \overline{m}Ep^{+}}{Var(p^{+})}, \\ b &= \frac{\left(\nu^{'}\Gamma^{-1}l - \overline{m}Ep^{++}\right)}{Cov(F_2, p^{++})} - a\frac{Cov(F_1, p^{++})}{Cov(F_2, p^{++})}, \\ c &= \frac{\left(\eta^{'}\left[\Gamma^{(2)}\right]^{-1}\eta - \overline{m}Ep^{*}\right)}{Cov(F_3, p^{*})} - a\frac{Cov(F_1, p^{*})}{Cov(F_3, p^{*})} - b\frac{Cov(F_2, p^{*})}{Cov(F_3, p^{*})}. \end{aligned}$$

The notation $EL[.|\mathcal{F}]$ indicates the fitted values from a linear regression on $\mathcal{F}$.

We use this SDF decomposition to provide an approach to portfolio selection.

3.2 Application to portfolio choice

It can be shown that the Markowitz approach to portfolio selection, which consists of minimizing the (unit cost) portfolio risk subject to the portfolio expected return, is based on factors p^{+} and p^{++}. Markowitz (1952) minimizes the portfolio risk subject to the portfolio cost and the expected payoff,

$$\begin{aligned} & \min_{p} \sigma\left(p\right), \\ \text{s.t. } Ep &= \mu_p,\, C\left(p\right) = 1. \end{aligned}$$

If p^{mv} denotes the optimal solution to the problem above, then it is the only linear combination of p^{+} and p^{++} satisfying the constraints. We consider an approach to portfolio selection based not only on p^{+} and p^{++} but also on p^{*}.

Definition 3.2 *Given the portfolio expected return, the cost of the squared primitive asset, η, and the cost of the squared portfolio return, c^{*}, the mean-variance-cost optimal portfolio is defined as the solution to the following program:*

$$\begin{aligned} & \min_{p} \;\; \sigma\left(p\right), \\ s.t. \; Ep &= \mu_p,\, C\left(p\right) = 1 \;,\, \widetilde{C}\left(p^2\right) = c^{*}, \end{aligned} \tag{3.6}$$

where $C\left(p\right)$ represents the cost of the portfolio p, and $\widetilde{C}\left(p^2\right)$ the cost of the squared portfolio return.

The difference between our optimization problem and the Markowitz optimization problem is that we minimize portfolio risk subject to an additional constraint that takes into account the portfolio skewness. We first solve (3.6) and then show the relationship between our approach to portfolio selection and the standard portfolio selection under skewness. If p^{mvs} denotes the optimal solution for problem (3.6), we have

$$p^{mvs} = \alpha_1 p^{+} + \alpha_2 F_2 + \alpha_3 F_3,$$

where α_1, α_2, and α_3 are determined by the equations below:

$$\begin{aligned} \alpha_1 Ep^{+} + \alpha_2 EF_2 + \alpha_3 EF_3 &= \mu_p, \\ \alpha_1 E\left(p^{+}p^{+}\right) + \alpha_2 E\left(F_2 p^{+}\right) + \alpha_3 E\left(F_3 p^{+}\right) &= 1, \\ \alpha_1 E\left(p^{+}p^{*}\right) + \alpha_2 E\left(F_2 p^{*}\right) + \alpha_3 E\left(F_3 p^{*}\right) &= c^{*}. \end{aligned}$$

The variance of p^{mvs} is

$$\sigma^2\left(c^{*}, \mu_p\right) = \alpha_1^2 Var\left(p^{+}\right) + \alpha_2^2 Var\left(F_2\right) + \alpha_3^2 Var\left(F_3\right). \tag{3.7}$$

To get the optimal portfolio weights, we have,

$$p^{mvs} = \omega_p^{'} R = R^{'} \omega_p.$$

Thus, premultiplying p^{mvs} by R and taking the expectation, we deduce

$$\omega_p = \alpha_1 \Gamma^{-1} E\left(Rp^+\right) + \alpha_2 \Gamma^{-1} E\left(RF_2\right) + \alpha_3 \Gamma^{-1} E\left(RF_3\right). \tag{3.8}$$

We refer to $\mathcal{E}_1$ as being the set

$$\mathcal{E}_1 = \left\{ \left(\mu_p, c^*, \sigma\left(p^{mvs}\right) : \left(\mu_p, c^*\right) \in \mathbb{R}^2\right) \right\},$$

where $\mathcal{E}_1$ represents the mean-variance-cost surface frontier. For each portfolio p^{mvs} in $\mathcal{E}_1$, we find the corresponding portfolio skewness $s_p = \frac{E\left(p^{mvs} - \mu_p\right)^3}{\sigma^3\left(c^*, \mu_p\right)}$. If we refer to $\mathcal{E}_2$ as being the set

$$\mathcal{E}_2 = \left\{ \left(\mu_p, s_p, \sigma\left(p^{mvs}\right) : \left(\mu_p, s_p\right) \in \mathbb{R}^2\right) \right\},$$

then $\mathcal{E}_2$ represents the mean-variance-skewness surface. Next, consider the payoff:

$$R^{mvs} = \frac{m^{mvs}\left(\eta, \overline{m}\right)}{C\left(m^{mvs}\left(\eta, \overline{m}\right)\right)}.$$

It follows that

$$C\left(R^{mvs}\right) = 1. \tag{3.9}$$

If c^*_{mvs} denotes the cost of $\left(R^{mvs}\right)^2$, it can be shown that

$$c^*_{mvs} = \left(\overline{m}^2 + \sigma^2\left[m^{mvs}\left(\overline{m}, \eta\right)\right]\right) E\left(R^{mvs}\right)^3.$$

Using (3.9), we show that

$$\frac{\left|1/\overline{m} - \mu_p\right|}{\sigma(p)} \leq \frac{\left|1/\overline{m} - ER^{mvs}\right|}{\sigma\left(R^{mvs}\right)} = \frac{\sigma\left[m^{mvs}\left(\overline{m}, \eta\right)\right]}{Em^{mvs}\left(\overline{m}, \eta\right)} \leq \frac{\sigma(m)}{Em} \; \forall m \in \mathcal{F}_2\left(\overline{m}, \eta\right). \tag{3.10}$$

Inequality (3.10) shows that no other portfolio with the same mean and same squared return cost has smaller variance than R^{mvs}. The return R^{mvs} belongs to the mean-variance-cost surface, $\mathcal{E}_1$.

Theorem 3.3 *R^{mvs} is mean-variance-cost efficient; i.e., no other portfolio with the same squared portfolio cost and the same mean has smaller variance.*

If we consider the return associated with the HJ minimum-variance SDF, which is:

$$R^{mv} = \frac{m_{HJ}\left(\overline{m}\right)}{C\left(m_{HJ}\left(\overline{m}\right)\right)},$$

it can be shown that

$$\frac{\left|1/\overline{m} - ER^{mv}\right|}{\sigma\left(R^{mv}\right)} = \frac{\sigma\left[m_{HJ}\left(\overline{m}\right)\right]}{Em_{HJ}\left(\overline{m}\right)}.$$

By proposition 2.1, we have:

$$\frac{\sigma\left[m_{HJ}\left(\overline{m}\right)\right]}{Em_{HJ}\left(\overline{m}\right)} < \frac{\sigma\left[m^{mvs}\left(\overline{m}, \eta\right)\right]}{Em^{mvs}\left(\overline{m}, \eta\right)}.$$

Therefore, the following inequality holds:

$$\frac{|1/\overline{m}-ER^{mv}|}{\sigma\left(R^{mv}\right)}<\frac{|1/\overline{m}-ER^{mvs}|}{\sigma\left(R^{mvs}\right)}.$$

The left-hand side of (3.10) represents the portfolio Sharpe ratio under the assumption that the risk-free return exists. If the risk-free return (R_F) exists (i.e., $R_F = 1/\overline{m}$), then R^{mvs} has a higher ratio than R^{mv}. In the light of this inequality and theorem 3.3, it is important to define in our setting the mean-variance-cost tangency portfolio.

Definition 3.4 *The mean-variance-cost tangency portfolio is the portfolio that has the maximum Sharpe ratio of all possible portfolios with identical squared portfolio cost.*

We next investigate how the portfolio skewness affects the squared portfolio cost. To see how this cost is a function of the portfolio skewness, consider the linear regression of p^2 on p,

$$p^2 = Ep^2 + \frac{Cov\left(p,p^2\right)}{Var\left(p\right)}\left(p-Ep\right)+\upsilon.$$

The cost of the squared portfolio return can be written as:

$$\begin{aligned} c^* &= Em^{mvs}p^2 \\ &= \overline{m}Ep^2+\left(1-\overline{m}\mu_p\right)\left[2\mu_p+\sigma_p s_p\right]+Cov\left(\upsilon,m^{mvs}\right), \end{aligned}$$

with $s_p = \frac{E\left(p-\mu_p\right)^3}{\sigma_p^3}$. Through this expression, the cost of the squared portfolio return is a function of the portfolio skewness. Therefore, it is reasonable to put forward the relationship between our approach to portfolio selection and the standard portfolio selection under skewness. The latter consists of minimizing the portfolio risk subject to the portfolio expected payoff and skewness. We formalize the standard approach to portfolio selection as follows:

$$\begin{aligned} &\min \quad \sigma\left(p\right), \\ \text{s.t.} \quad & C\left(p\right)=1 \\ & Ep=\mu_p \\ & \frac{E\left(p-\mu_p\right)^3}{\sigma_p^3}=s_p, \end{aligned} \tag{3.11}$$

where s_p represents the portfolio skewness. Apart from the two portfolio constraints (expected return and portfolio cost), it can be observed that the difference between our optimization problem and standard portfolio selection under skewness comes from the third constraint. In standard portfolio selection under skewness, the third constraint is on the portfolio skewness, whereas, in our approach, the third constraint is on the cost of the squared portfolio return. De Athayde and Flores (2004) find a general solution to problem (3.11). It is thus important to study the relationship between the two problems in (3.6) and (3.11). We will say that problems (3.6) and (3.11) are observationally equivalent if and only if any optimal solution to problem (3.6) is also optimal to problem (3.11), and vice versa.

We derive necessary and sufficient conditions that make our approach to portfolio selection observationally equivalent to standard portfolio selection under skewness.

Proposition 3.5 *Consider a portfolio p and the linear regression of p^2 on p:*

$$p^2 = EL\left[p^2|p\right] + \upsilon.$$

Then, $Cov\left(\upsilon, m^{mvs}\right) = 0$ for all portfolios p if and only if the components of the price, $\eta = \overline{\eta}$, of $R^{(2)}$ are:

$$\overline{\eta}_{ii} = \overline{m}ER_i^2 + \left(1 - \overline{m}ER_i\right)\left[2ER_i + \frac{E\left(R_i - ER_i\right)^3}{Var\left(R_i\right)}\right] \text{ for } i\text{=1,...,}n,$$

and

$$\overline{\eta}_{ij} = \frac{1}{2}\overline{m}\left(ER_i^2 + ER_j^2 + 2ER_iR_j\right) + \frac{\left[1 - \frac{1}{2}\left(ER_i + ER_j\right)\overline{m}\right] Cov\left(\left(R_i + R_j\right), \left(R_i + R_j\right)^2\right)}{\left[Var\left(R_i\right) + Var\left(R_j\right) + 2Cov\left(R_i, R_j\right)\right]} - \frac{1}{2}\left(\overline{\eta}_{ii} + \overline{\eta}_{jj}\right)$$

for $i \neq j$.

Proof. See the proof in the appendix. ■

Proposition 3.6 gives the necessary and sufficient conditions to get the standard portfolio selection under skewness; that is, a maximum skewness portfolio (see de Athayde and Flores 2004).

Proposition 3.6 *If $\mu_p \neq 1/\overline{m}$, consider a portfolio p and the linear regression of p^2 on p:*

$$p^2 = EL\left[p^2|p\right] + \upsilon.$$

Problems (3.11) and (3.6) are observationally equivalent if and only if $Cov\left(\upsilon, m^{mvs}\right) = 0$ for any portfolio p.

Proof. If $Cov\left(\upsilon, m^{mvs}\right) = 0$, we have,

$$c^* = \overline{m}\left(\sigma_p^2 + \mu_p^2\right) + \left(1 - \overline{m}\mu_p\right)\left[2\mu_p + \sigma_p s_p\right]. \tag{3.12}$$

This equation is equivalent to

$$\overline{m}\sigma_p^2 + \sigma_p s_p\left(1 - \overline{m}\mu_p\right) + \left(2\mu_p - \overline{m}\mu_p^2 - c^*\right) = 0.$$

Using (3.12), it is obvious that $\left\{p\text{: } Ep = \mu_p \text{ and } \frac{E\left(p - \mu_p\right)^3}{\sigma_p^3} = s_p\right\}$ and $\left\{p\text{: } Ep = \mu_p \text{ and } \widetilde{C}\left(p^2\right) = c^*\right\}$ are equivalent. Therefore, problems (3.11) and (3.6) are observationally equivalent.

In other respects, assume that (3.6) and (3.11) are observationally equivalent. Thus, they produce an identical solution. This implies that problem (3.11) can be used to compute the cost of the squared portfolio return. This is possible only if $Cov\left(\upsilon, m^{mvs}\right) = 0$. ■

This proposition shows how our approach to portfolio selection generalizes standard portfolio selection under skewness and suggests that standard portfolio selection under skewness implicitly assumes that the covariance of m^{mvs} with υ is null for any portfolio p.

We assume that μ_p and s_p are known and use a simple methodology to get a maximum skewness portfolio solution to problem (3.11):

- First, under the assumption $Cov\left(\upsilon, m^{mvs}\right) = 0$, we compute the cost of $R^{(2)}$, $\eta = \overline{\eta}$, using proposition 3.5.

- Second, given the portfolio skewness and expected return, we compute c^* as follows: the discriminant of equation (3.12) is

$$\Delta = s_p^2\left(1 - \overline{m}\mu_p\right)^2 - 4\overline{m}\left(2\mu_p - \overline{m}\mu_p^2 - c^*\right).$$

Assuming that $\Delta > 0$, this equation produces two solutions:

$$\sigma_p = \frac{-s_p\left(1 - \overline{m}\mu_p\right) - \sqrt{\Delta}}{2\overline{m}} \text{ or } \sigma_p = \frac{-s_p\left(1 - \overline{m}\mu_p\right) + \sqrt{\Delta}}{2\overline{m}}. \tag{3.13}$$

From (3.13),

$$\sigma_p^2\left(c^*, \mu_p\right) = \frac{\left[s_p\left(1 - \overline{m}\mu_p\right) + \sqrt{\Delta}\right]^2}{\left(2\overline{m}\right)^2} \text{ or } \sigma_p^2\left(c^*, \mu_p\right) = \frac{\left[s_p\left(1 - \overline{m}\mu_p\right) - \sqrt{\Delta}\right]^2}{\left(2\overline{m}\right)^2}. \tag{3.14}$$

If ($s_p > 0$ and $1-\overline{m}\mu_p > 0$) or ($s_p < 0$ and $1-\overline{m}\mu_p < 0$), then, according to (3.7), the minimum-variance portfolio is:

$$\sigma_p^2\left(c^*, \mu_p\right) = \alpha_1^2 Var\left(p^+\right) + \alpha_2^2 Var\left(F_2\right) + \alpha_3^2 Var\left(F_3\right) = \frac{\left[s_p\left(1 - \overline{m}\mu_p\right) - \sqrt{\Delta}\right]^2}{\left(2\overline{m}\right)^2}, \tag{3.15}$$

with

$$\begin{aligned} \alpha_1 &= 1/C\left(p^+\right), \\ \alpha_2 &= A_1 - A_2 c^*, \\ \alpha_3 &= A_3 c^* - A_4, \end{aligned}$$

where A_1, A_2, A_3, and A_4 are known parameters. Equation (3.15) is equivalent to

$$\begin{aligned} \frac{\left[s_p\left(1 - \overline{m}\mu_p\right) - \sqrt{\Delta}\right]^2}{\left(2\overline{m}\right)^2} &= \alpha_1^2 Var\left(p^+\right) + \left[A_1^2 + A_2^2 c^{*2} - 2A_1A_2c^*\right] Var\left(F_3\right) + \\ & \left[A_3^2 + A_4^2 c^{*2} - 2A_4A_3c^*\right] Var\left(F_3\right). \end{aligned} \tag{3.16}$$

This equation can be rewritten in terms of Δ. There might be more than one solution to this equation. Choose the solution Δ that yields a smaller variance and use this Δ to compute c^*. The same methodology can be repeated if ($s_p < 0$ and $1 - \overline{m}\mu_p > 0$) or ($s_p > 0$ and $1 - \overline{m}\mu_p < 0$).

- Once c^* is computed, (3.16) gives the minimum variance to problem (3.11).

In the next section, we illustrate the portfolio selection and investigate empirically whether $Cov\left(v, m^{mvs}\right) \neq 0$.

4. Portfolio Selection: An Empirical Illustration

To give an empirical illustration of our approach to portfolio selection, we need to know the cost of the squared primitive assets. To compute this cost, we assume that the joint process of the SDF and asset returns is lognormal, since this distribution is more flexible and allows for skewness. It is often used to characterize asset probability models. For example, many asset-pricing tests assume that the joint process of the SDF-asset returns is conditionally lognormal. Diffusion models imply a locally lognormal distribution. The next proposition gives the cost of the squared primitive assets when the joint process of the SDF and asset returns is lognormal.

Proposition 4.1 *Given an SDF m, consider a set of N primitive assets. Assume that the joint process $\left(Log\left(m\right), Log\left(R\right)\right)$ follows a multivariate normal distribution. Thus, the components of η are of the form:*

$$\begin{aligned} \eta_{ij} &= E\left(mR_iR_j\right) \\ &= \frac{1}{\overline{m}} \frac{ER_iR_j}{ER_iER_j} \ \forall i, j. \end{aligned}$$

Proof. See the proof in the appendix. ■

To gauge the empirical importance of the cost of the squared portfolio in portfolio selection, we collect daily asset returns from the Datastream data base for the sample period from January 2002 to June 2002. This data set consists of the daily returns of four highly liquid stocks: General Motors, Cisco Systems, Boeing, and Ford Motors. Over the same period, we extract the U.S. 3-month Treasury bill rate (risk-free rate). The estimated U.S 3-month Treasury bill expected return is 1.0495. Table 1 reveals that Boeing has the lowest expected return and highest positive skewness, while Cisco Systems has a negative skewness. We use (3.6) to find the optimal portfolio. Figure 8 illustrates the mean-variance-cost surface, $\mathcal{E}_1$, and the associated mean-variance-skewness surface, $\mathcal{E}_2$. Slicing the surface at any level of squared portfolio cost, we get the familiar positively sloping portion of the mean-variance frontier. In the standard mean-variance analysis there is a single efficient risky-asset portfolio, but in our setting there are multiple efficient portfolios. The mean-variance-skewness surface reveals that the squared portfolio cost and the portfolio skewness have a significant impact on the portfolio mean-variance frontier. This is shown more clearly in Figure 10; the figure shows how small changes in the cost of the squared portfolio return have a great impact on the portfolio mean-variance frontier. This indicates that the cost of the squared portfolio will significantly impact the tangency portfolio. Note that, at any level of the squared portfolio cost, we get the positively sloping portion of the mean-variance frontier. But at any level of the portfolio skewness (see the mean-variance-skewness surface), we do not have the usual, positively sloping portion of the mean-variance frontier. This intuitively shows that our approach is not observationally equivalent to standard portfolio selection under skewness. From proposition 3.6, however, our approach is observationally equivalent to standard portfolio selection under skewness when $Cov\,(m^{mvs}, \upsilon) = 0$. Figure 9 illustrates the mean-variance-skewness surface when $Cov\,(m^{mvs}, \upsilon) = 0$. Figure 9 shows that, at any level of the portfolio skewness, varying the portfolio mean produces the usual positively sloping portion of the mean-variance frontier.[3] Figure 11 illustrates the implied covariance of the SDF with the residuals obtained when regressing the squared portfolio on the portfolio itself. Figure 11 provides empirical evidence that this covariance is different from zero and negative.

5. Conclusions

In this paper, we have derived a new variance bound on any admissible SDF that prices correctly a set of primitive assets and quadratic payoffs of the same primitive assets. Our bound tightens the HJ bound by an additional component that is a function of the squared primitive asset cost and asset co-skewness. We have given the necessary and sufficient conditions to get the well-known HJ bound. Using the Standard & Poor's 500 stock index and commercial paper rate from 1889 to 1994, we have illustrated our volatility bound and shown empirically that, when the SDF mean is in the neighbourhood of 1, our variance bound is twice as large as the HJ bound. We have also found that the SDF implied from the consumption-based models, such as Epstein and Zin's (1989) state non-separable preferences model, passes the HJ bound for a particular value of the relative risk-aversion coefficient, but does not pass our variance bound, making the equity-premium puzzle even more difficult to solve.

Motivated by the duality between the HJ bound and the Markowitz mean-variance analysis, we have offered an approach to portfolio selection based on factors that span our minimum-variance SDF. We have shown that our approach to portfolio selection generalizes standard portfolio selection under skewness, which consists of minimizing the portfolio risk subject to the portfolio expected payoff and skewness. We have used daily asset returns to illustrate our findings empirically. For the purposes of our illustration, we have assumed that the joint process of the SDF and asset returns is lognormal. This has allowed us to compute the cost of the squared primitive asset and then illustrate our approach to portfolio selection. Empirical results suggest that the cost of the squared portfolio return and the portfolio skewness have a significant impact on the portfolio mean-variance frontier.

Since Bekaert and Liu (2004) and others use conditional information to tighten the HJ bound, it would be of interest to examine how conditional information might be used to tighten our variance bound. In

[3] Gamba and Rossi (1998a, b) and de Athayde and Flores (2004) illustrate the mean-variance-skewness surface by solving problem (3.11).

light of Hansen and Jagannathan (1997), it appears natural to develop an SDF-based distance measure for asset-pricing models under this higher-order moments framework. We leave these issues for future research.

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Table 1: Asset Returns of Four Highly Liquid Stocks

(January 2002 to June 2002)

Asset company	Portfolio variable ω	Expected return	Variance	Skewness
General Motors	ω_1	1.0011	$0.2853e^{-3}$	0.2835
Cisco Systems	ω_2	1.0044	$0.3938e^{-3}$	-0.1244
Boeing	ω_3	0.9999	$0.3621e^{-3}$	0.6637
Ford Motors	ω_4	1.0049	$0.3777e^{-3}$	0.5045

Note: The skewness is measured by the third central moment divided by the cube of the standard deviation.

Figure 1: **SDF Volatility Surface Frontier with a Single Excess Return**

We use our approach to compute a mean-standard deviation-cost surface for SDFs using the excess simple return of the Standard & Poor's 500 stock index over the commercial paper rate. Annual U.S. data, from 1889 to 1994, are used to compute the SDF variance bound. The SDF feasible region is above this surface.

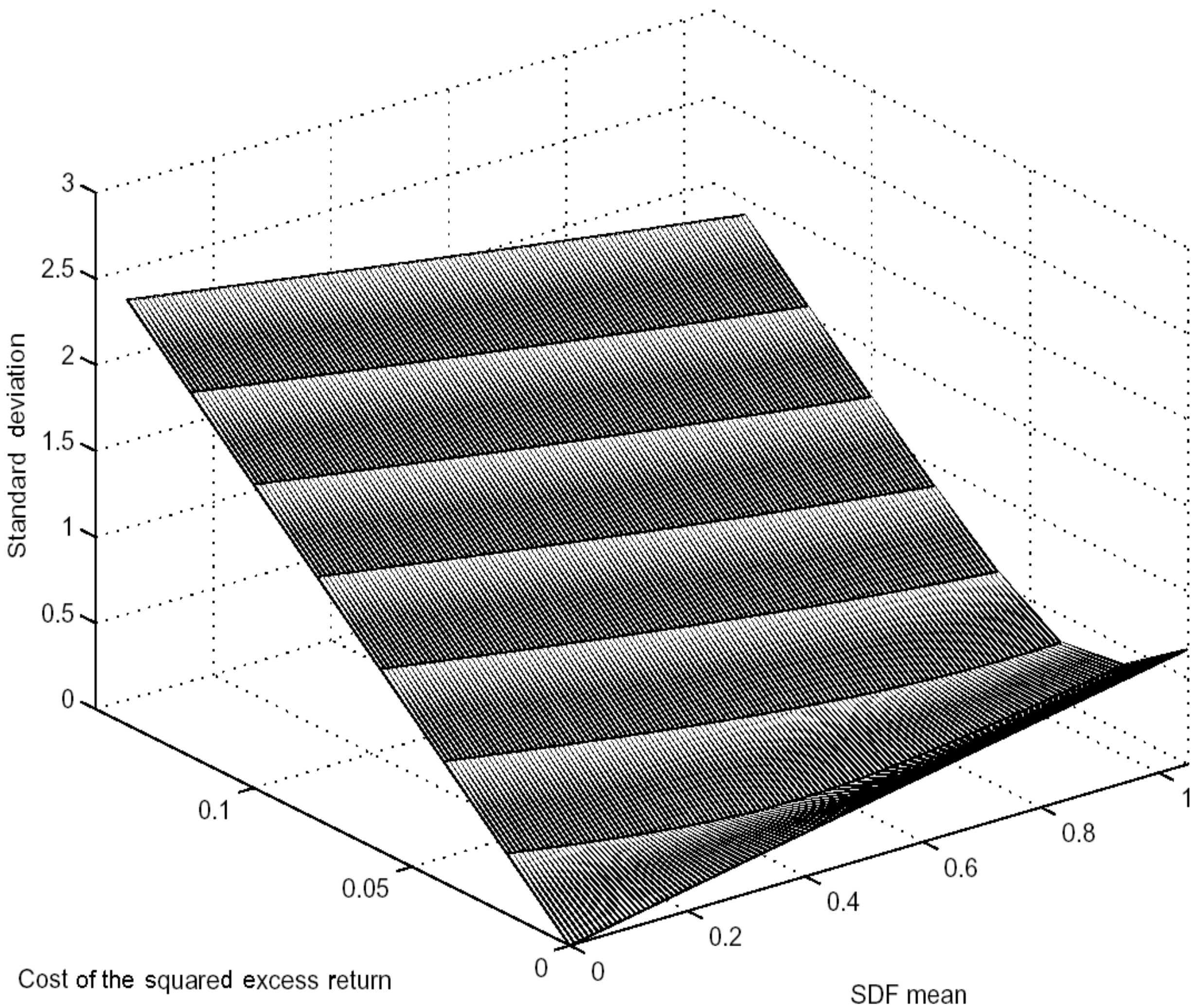


Figure 2: **HJ Frontier with a Single Excess Return**

We use the HJ approach to compute a standard deviation-mean frontier for SDFs using the excess simple return of the Standard & Poor's 500 stock index over the commercial paper rate. Annual data from 1889 to 1994 are used to plot this frontier. The SDF feasible region is above this frontier.

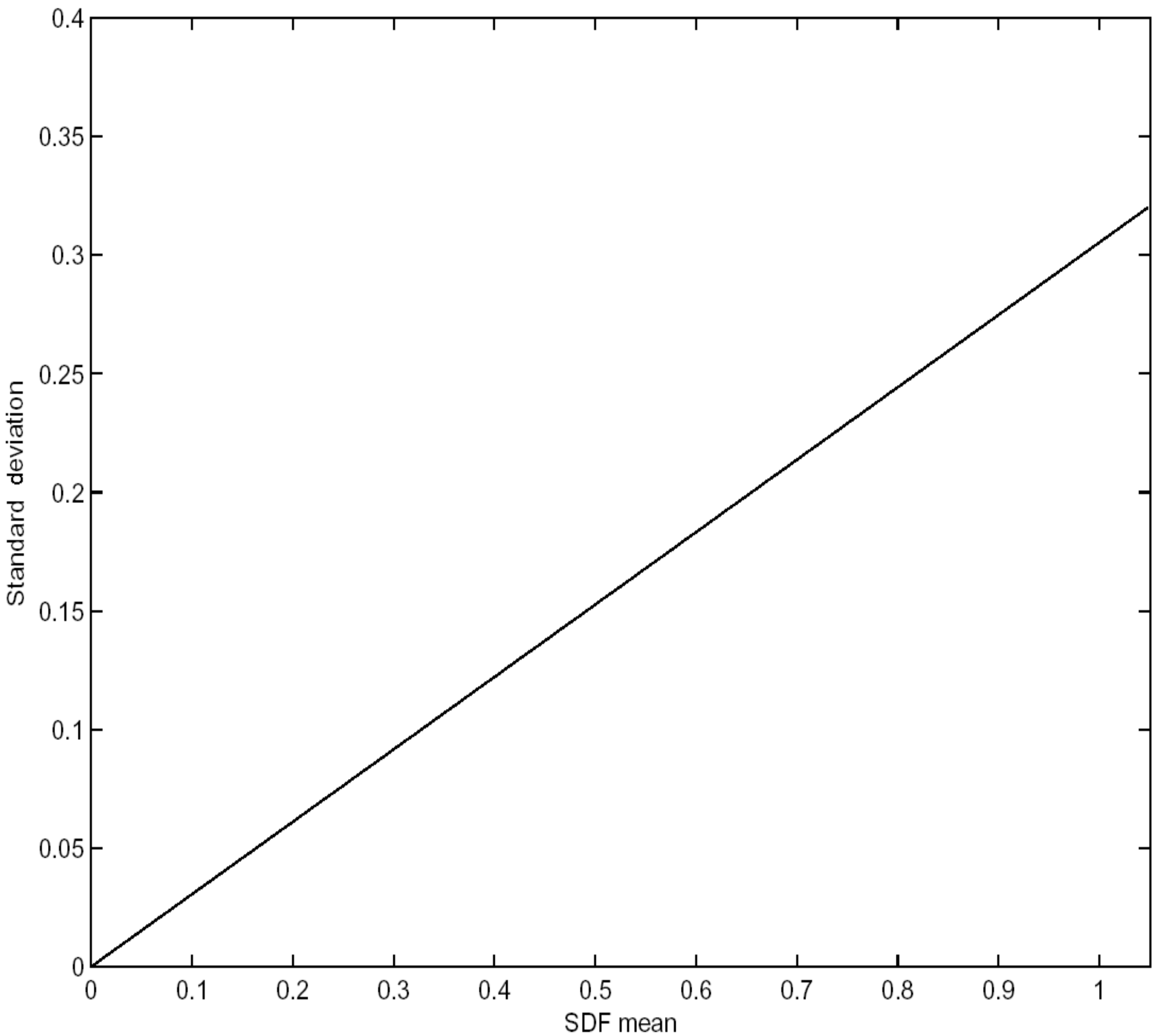


Figure 3: **HJ Volatility Frontier**

We imply a mean standard-deviation frontier for SDFs using the return of the Standard & Poor's 500 stock index and the commercial paper rate. Annual U.S. data, from 1889 to 1994, are used to compute the HJ variance bound. The SDF feasible region is above this frontier. With CRRA preferences, we vary exogenously the relative risk-aversion coefficient and trace out the resulting pricing kernels in this two-dimensional space. These pricing kernels are represented by asterisks (*). The first asterisk on the x-axis represents the implied pricing kernel for $\alpha = 0$. The last asterisk represents $\alpha = 25$.

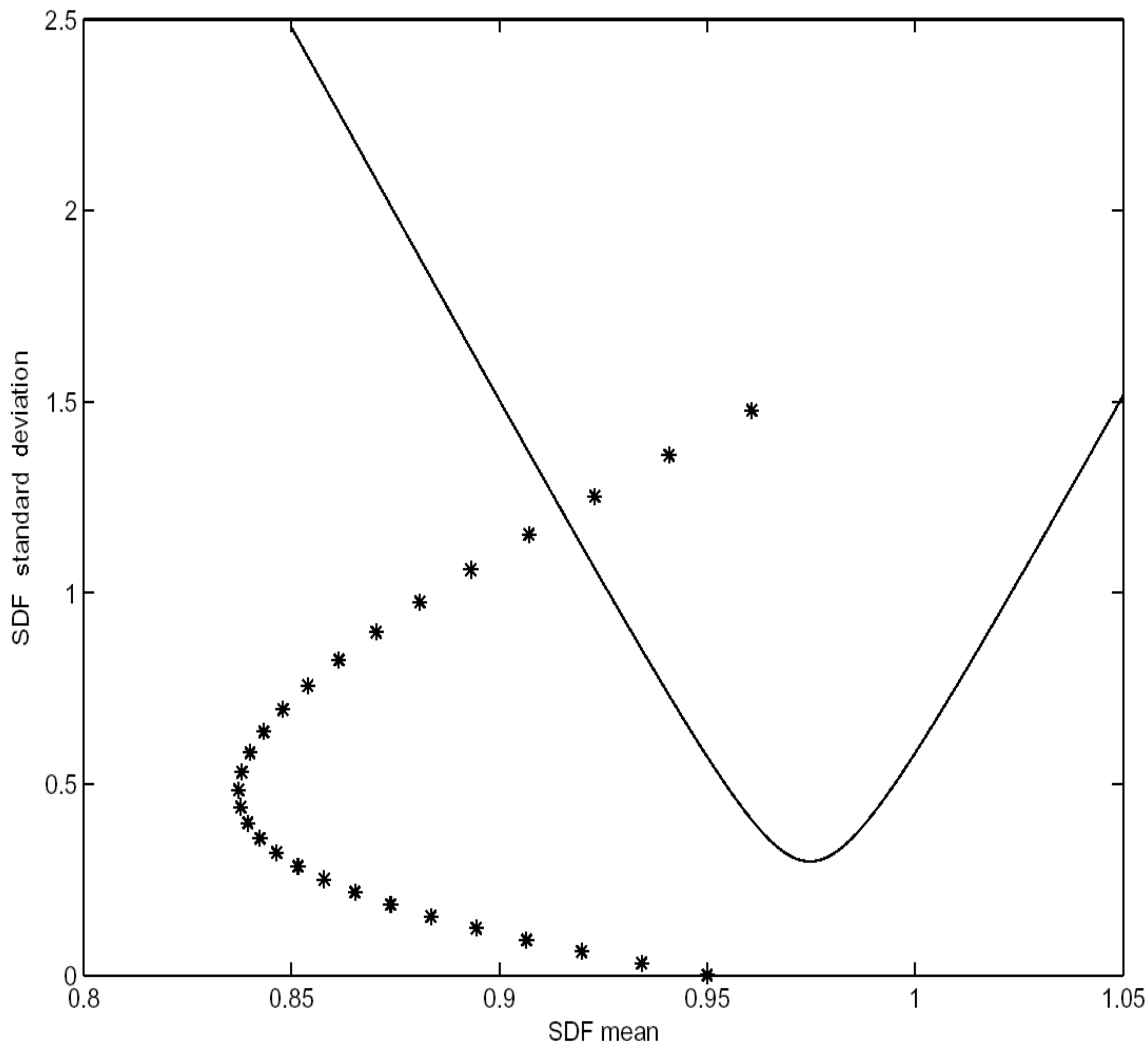


Figure 4: **SDF Volatility Frontier**

HJ represents the Hansen and Jagannathan volatility frontier and CY represents our volatility frontier. For each α, we find $\eta = EmR^{(2)}$ and trace out the point $(\overline{m}, \sigma(m^{mvs}(\overline{m}, \eta)))$ in a two-dimensional space. We also plot the point $(Em_{t+1}, \sigma(m_{t+1}))$ where m_{t+1} represents the SDF obtained in the investor optimization problem with CRRA preferences. We use the return of the Standard & Poor's 500 stock index over the commercial paper rate. Annual U.S. data, from 1889 to 1994, are used to compute the SDF variance bound.

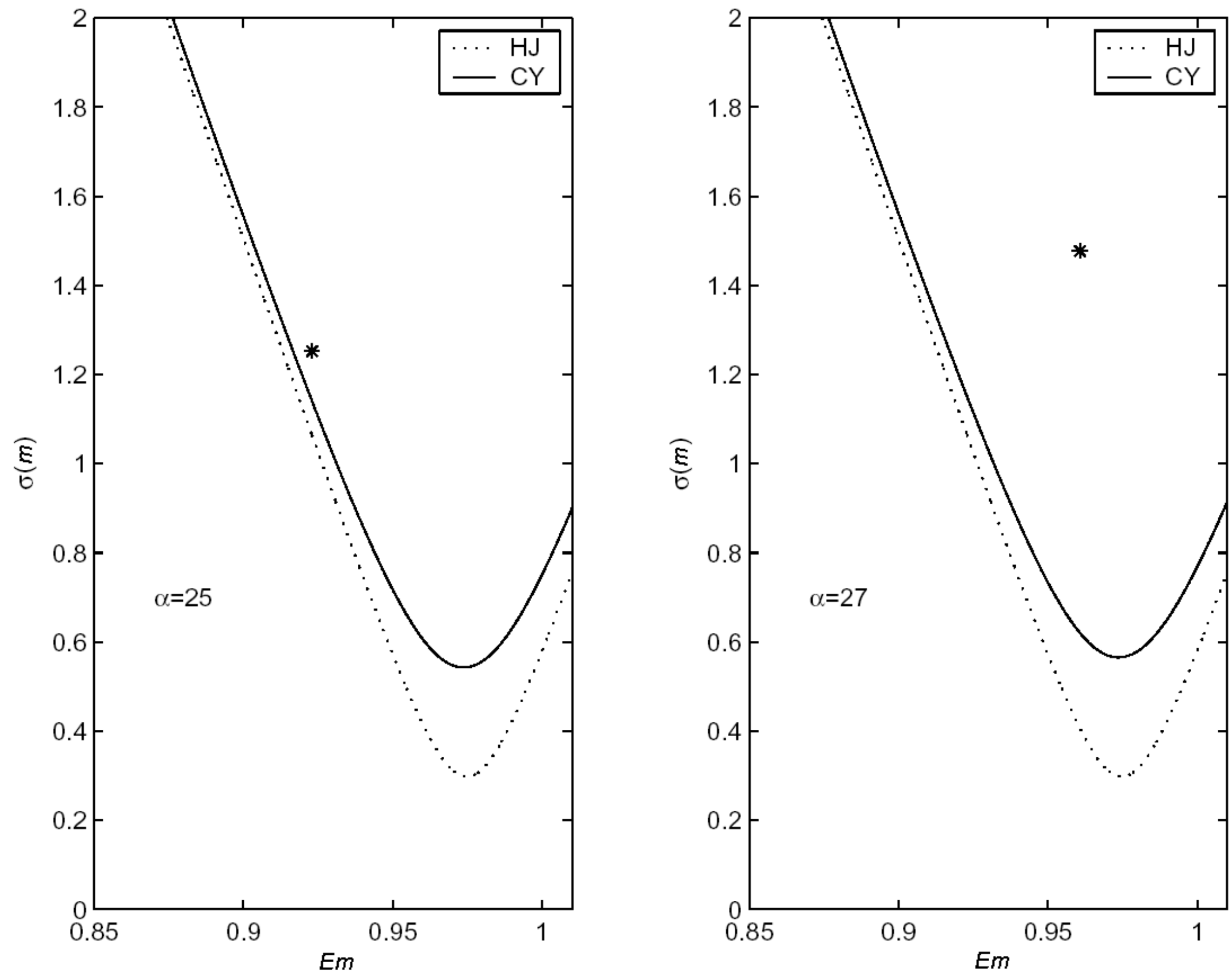


Figure 5: **SDF Volatility Frontier**

HJ represents the Hansen and Jagannathan volatility frontier and CY represents our volatility frontier. For each (α, ρ), we find $\eta = EmR^{(2)}$ and trace out the point $(\overline{m}, \sigma(m^{mvs}(\overline{m}, \eta)))$ in a two-dimensional space. We also plot the point $(Em_{t+1}, \sigma(m_{t+1}))$ where m_{t+1} represents the SDF obtained with Epstein and Zin (1989) state non-separable preferences. We use the return of the Standard & Poor's 500 stock index over the commercial paper rate. Annual U.S. data, from 1889 to 1994, are used to compute the SDF variance bound.

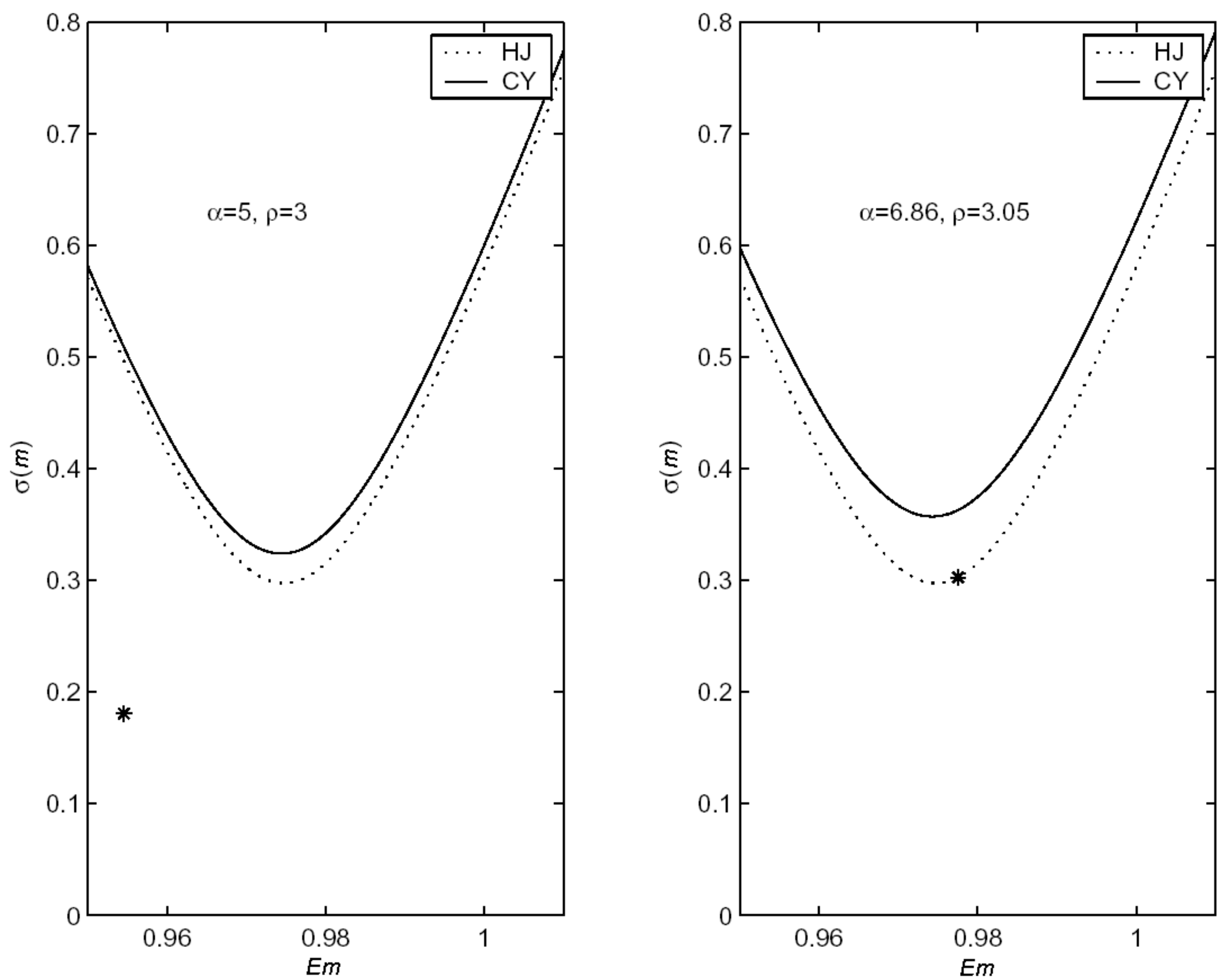


Figure 6: **SDF Volatility Frontier**

HJ represents the Hansen and Jagannathan volatility frontier and CY represents our volatility frontier. For each $(\theta, \alpha(1), \alpha(2))$, we find $\eta = EmR^{(2)}$ and trace out the point $(\overline{m}, \sigma(m^{mvs}(\overline{m}, \eta)))$ in a two-dimensional space. We also plot the point $(Em_{t+1}, \sigma(m_{t+1}))$ where m_{t+1} represents the SDF obtained with Gordon and St-Amour (2000) state-dependent preferences. We use the return on the Standard & Poor's 500 stock index over the commercial paper rate. Annual U.S. data, from 1889 to 1994, are used to compute the SDF variance bound.

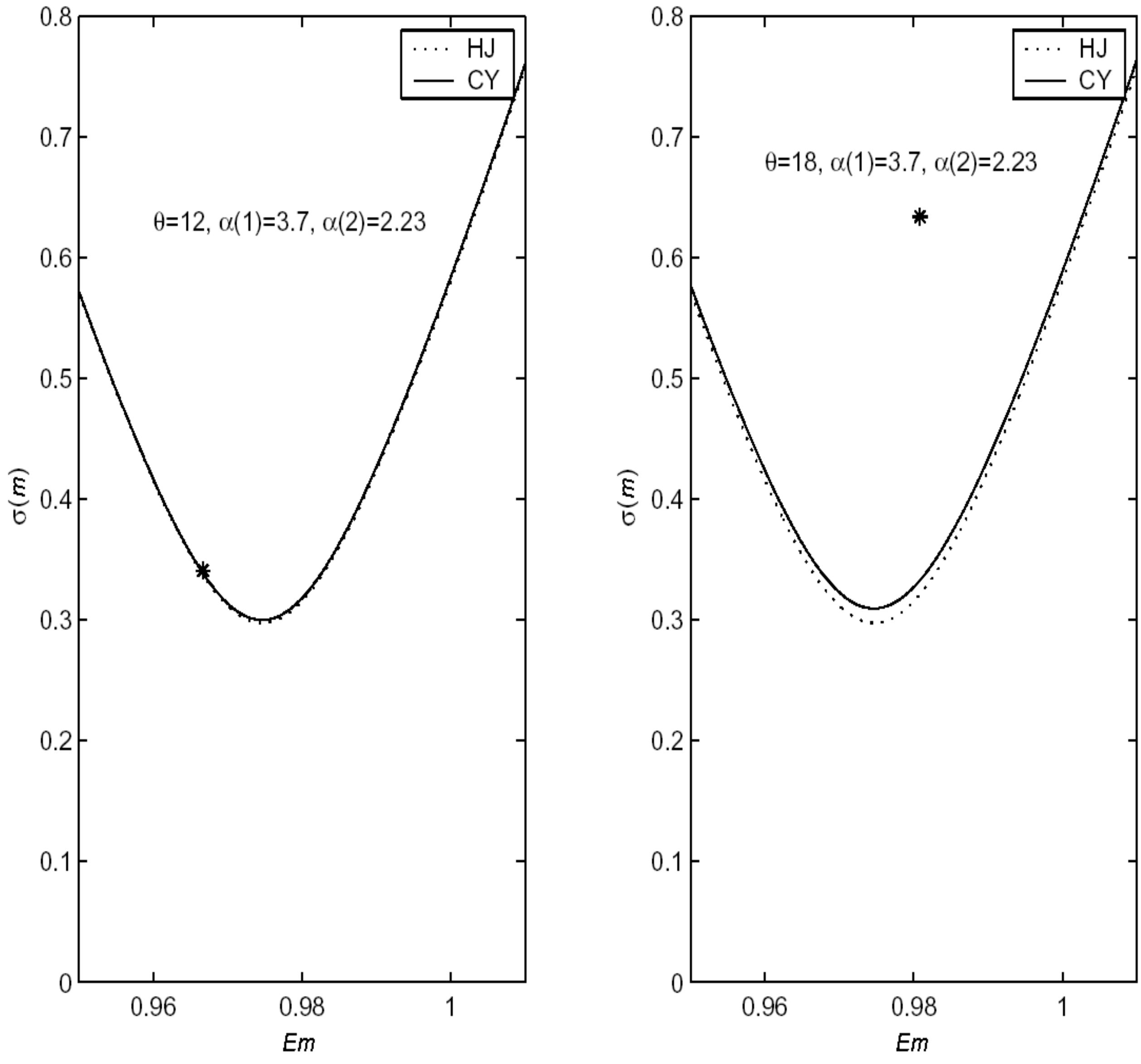


Figure 7: **SDF Volatility Frontier**

HJ represents the Hansen and Jagannathan volatility frontier and CY represents our volatility frontier. For each $(\alpha(1), \alpha(2), \rho)$, we find $\eta = EmR^{(2)}$ and trace out the point $(\overline{m}, \sigma(m^{mvs}(\overline{m}, \eta)))$ in a two-dimensional space. We also plot the point $(Em_{t+1}, \sigma(m_{t+1}))$ where m_{t+1} represents the SDF obtained with Melino and Yang (2003) state-dependent preferences with constant EIS, constant β, and state-dependent CRRA. We use the return of the Standard & Poor's 500 stock index over the commercial paper rate. Annual U.S. data, from 1889 to 1994, are used to compute the SDF variance bound.

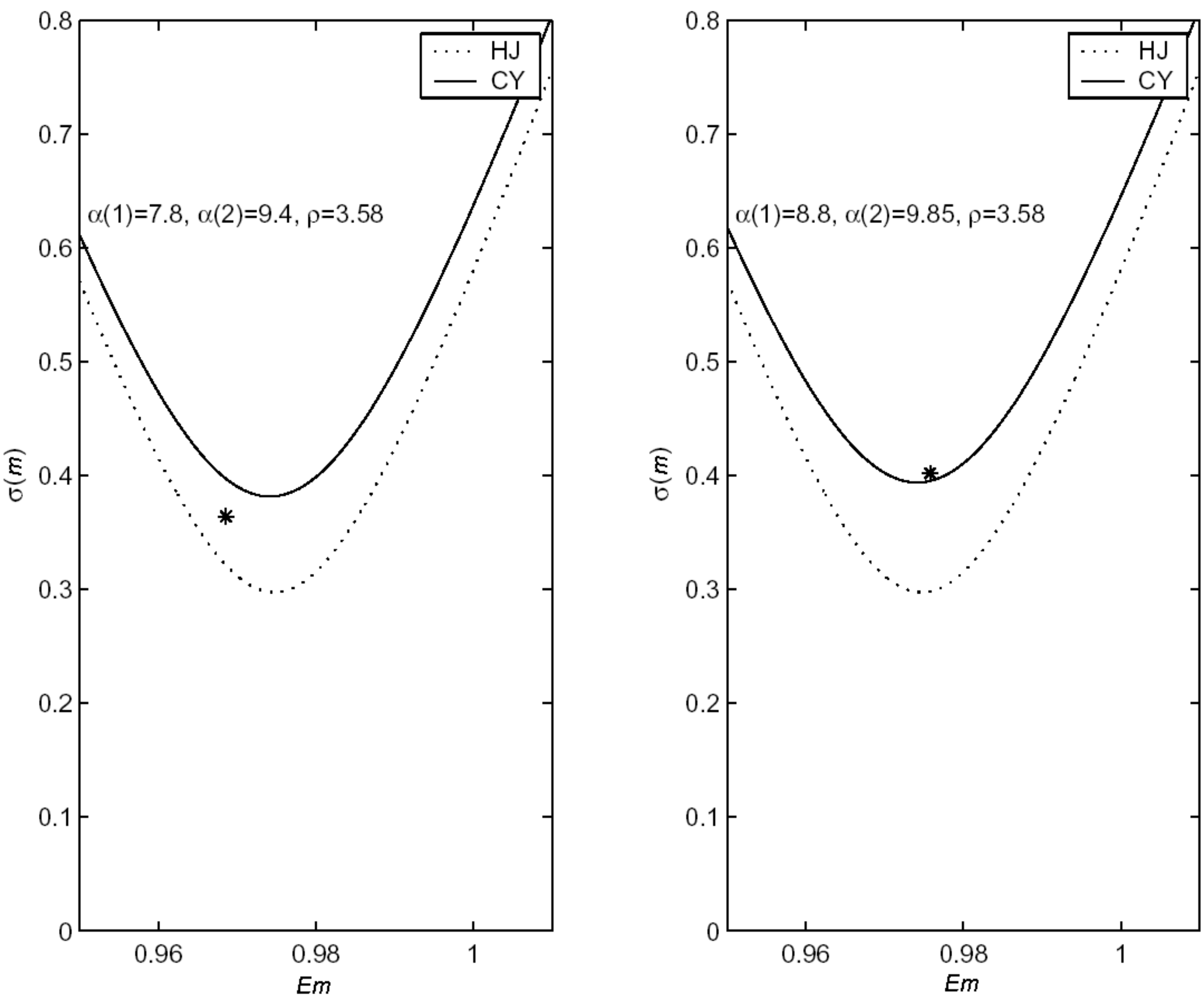


Figure 8: **Mean-Variance-Cost (M-V-C) and Mean-Variance-Skewness (M-V-S) Surfaces**
Given the portfolio mean, μ_p, and squared portfolio cost, c^*, we solve problem (3.6) and plot in a three-dimensional space the optimal portfolio $\left(\mu_p, c^*, \sigma\left(p^{mvs}\right)\right)$. Then we vary exogenously μ_p and c^* and get the M-V-C surface. We thereafter plot each optimal portfolio in a three-dimensional space: mean-standard deviation-skewness (see the M-V-S surface).

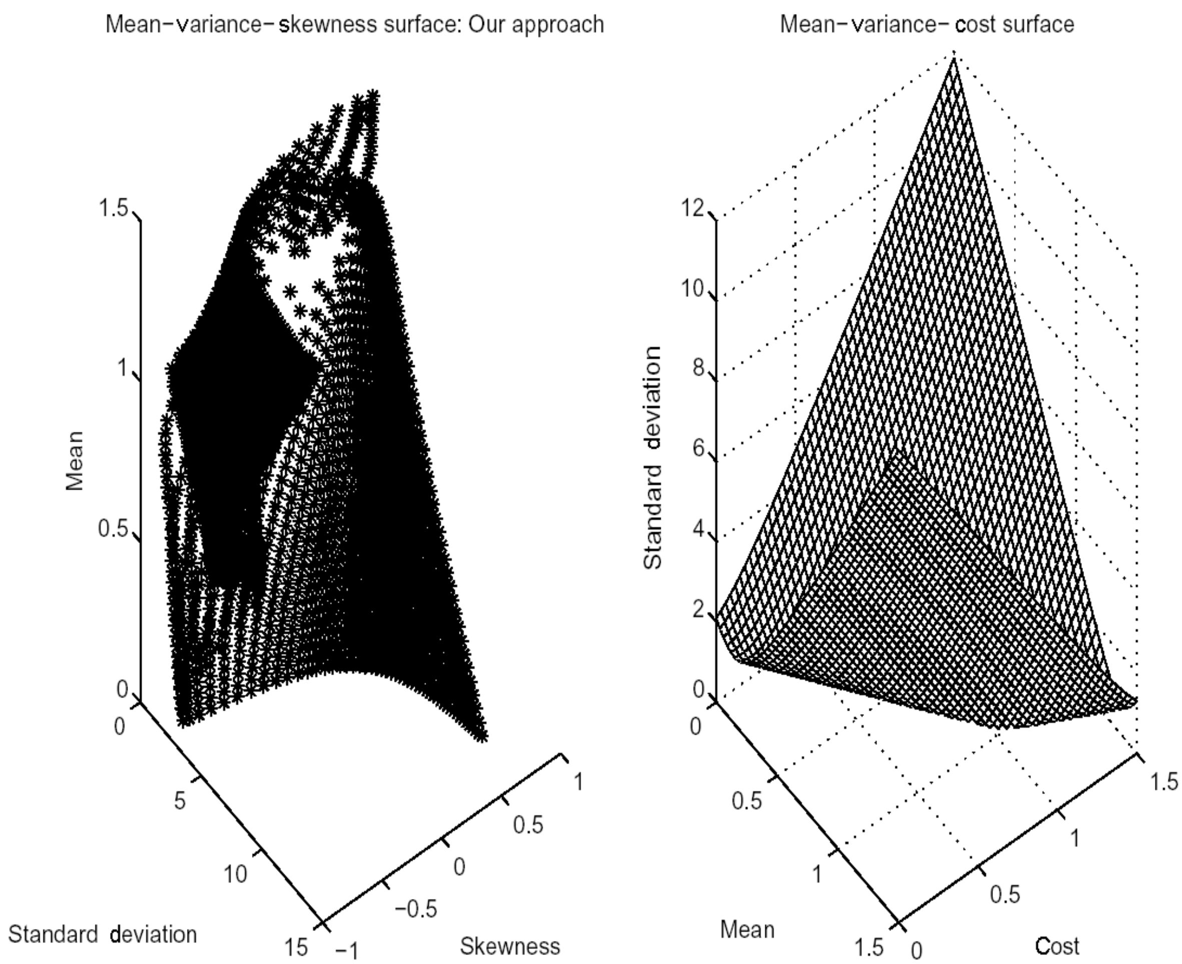


Figure 9: **Mean-Variance-Cost (M-V-C) and Mean-Variance-Skewness (M-V-S) Surfaces**

We assume $Cov\left(m^{mvs}, v\right) = 0$. Given the portfolio mean, μ_p, and squared portfolio cost, c^*, we solve problem (3.6) and plot in a three-dimensional space the optimal portfolio $\left(\mu_p, c^*, \sigma\left(p^{mvs}\right)\right)$. Then we vary exogenously μ_p and c^* and get the M-V-C surface. We thereafter plot each optimal portfolio in a three-dimensional space: mean-standard deviation-skewness (see the M-V-S surface).

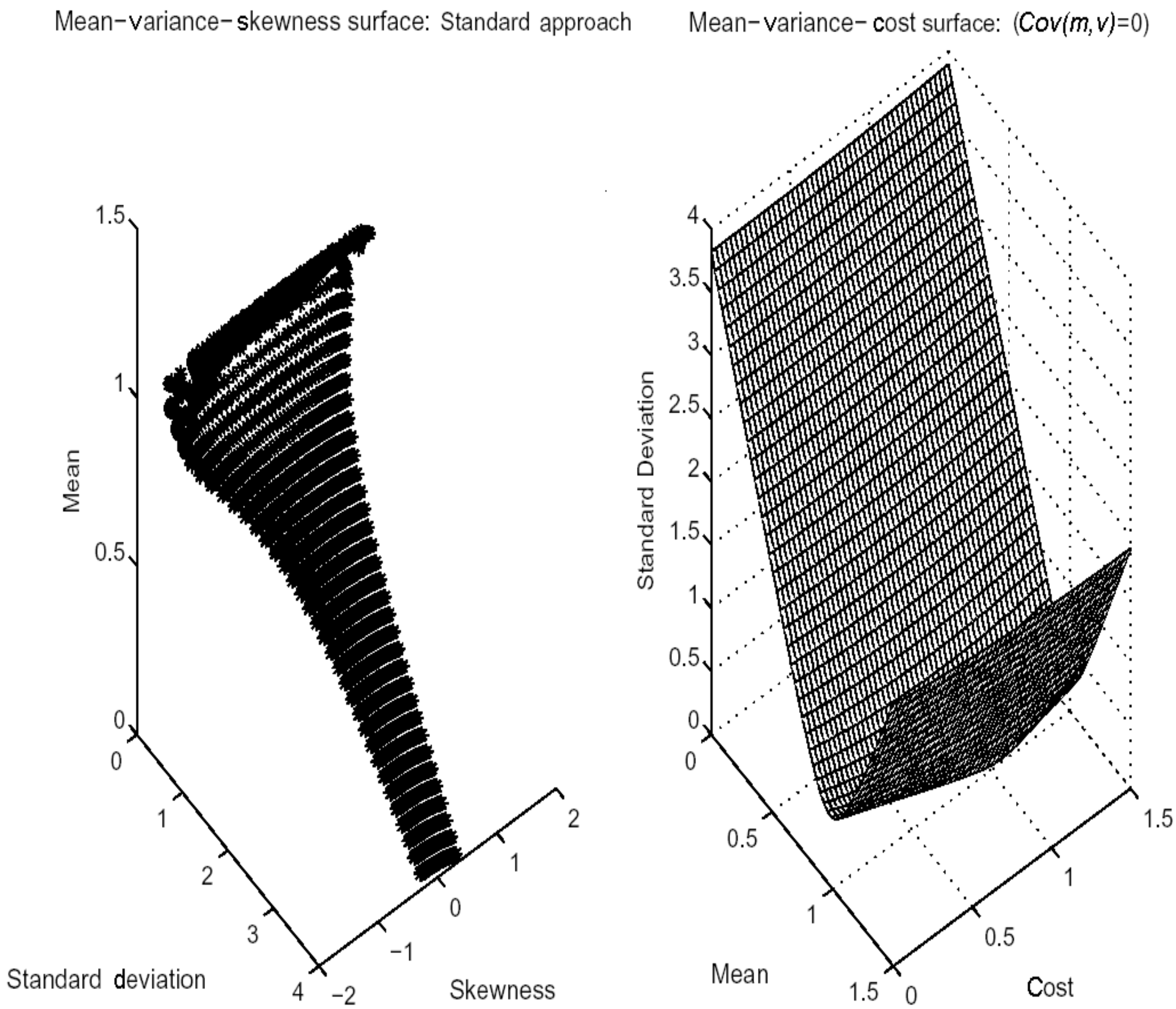


Figure 10: **Mean-Variance Frontier**

We first plot the Markowitz mean-variance (M-V) portfolio frontier, and then our mean-variance portfolio frontier (CY) for c^* =0.95 and 1.

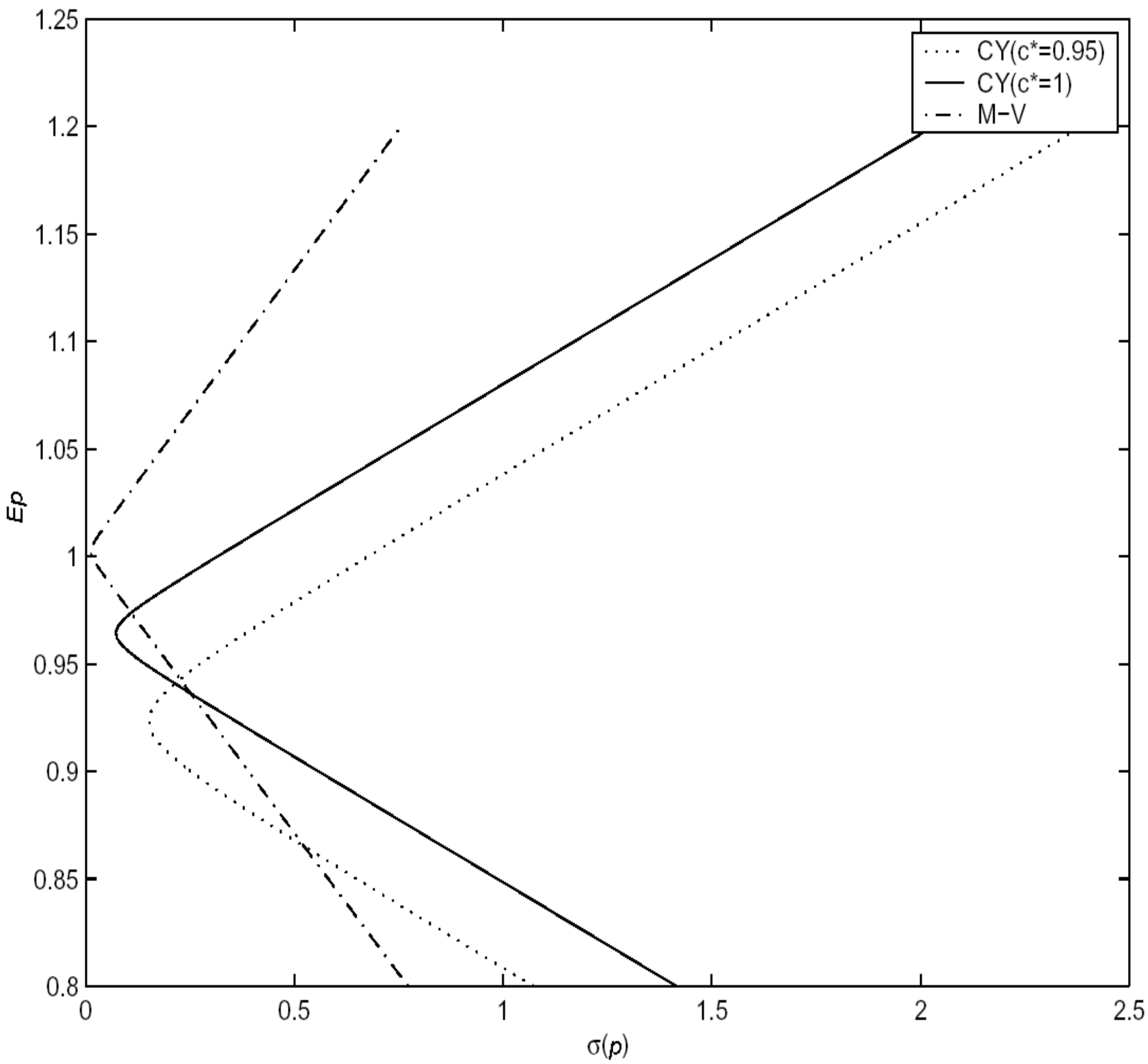
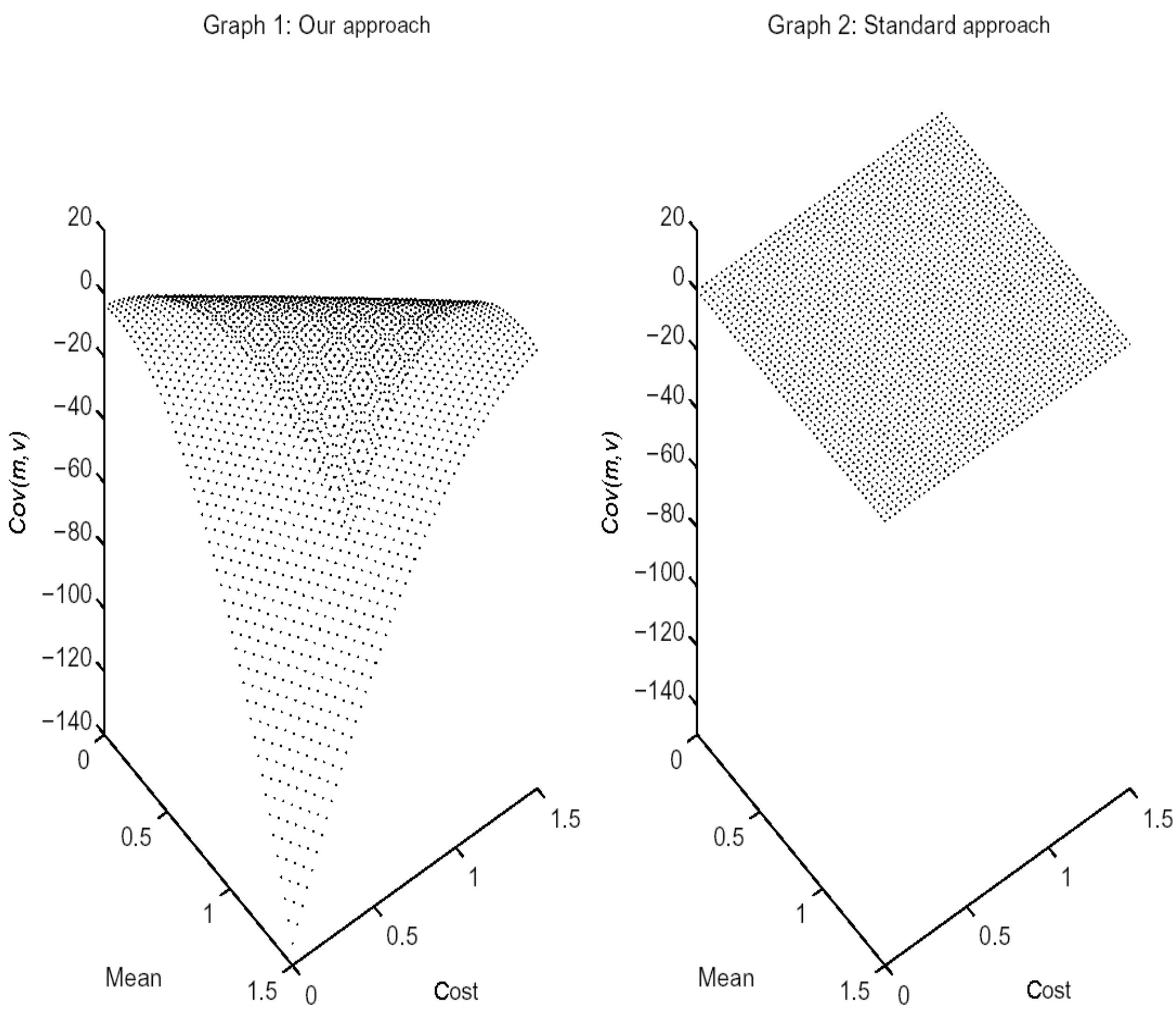


Figure 11: For each portfolio p that belongs to the mean-variance-cost surface, $\mathcal{E}_1$ (see Figures 8 and 9), we plot within Graph 1 the point $\left(\mu_p, Cov\left(m^{mvs}, v\right), c^*\right)$ where μ_p represents the portfolio mean, c^* is the cost of the squared portfolio return, and $Cov\left(m^{mvs}, v\right)$ is the covariance of the SDF with the residuals obtained when regressing the squared portfolio on the portfolio itself. Graph 2 represents this covariance when the standard portfolio selection under skewness is used.



Appendix

Proof of Theorem 3.1. If $\eta = \eta^*$, $m^{mvs}(\eta, \overline{m})$ collapses to the HJ stochastic discount factor. Let us assume that $\eta \neq \eta^*$ and that the minimum-variance SDF $m^{mvs}(\eta, \overline{m})$ can be decomposed as:

$$m^{mvs}(\eta, \overline{m}) = \overline{m} + aF_1 + bF_2 + cF_3,$$

where

$$\begin{aligned} F_1 &= p^+ - Ep^+, \\ F_2 &= p^{++} - EL\left[p^{++}|F_1\right], \\ F_3 &= p^* - EL\left[p^*|F_1, F_2\right]. \end{aligned}$$

First,

$$Cov\left(m^{mvs}(\eta, \overline{m}),\ p^+\right) = aCov\left(F_1, p^+\right) = aVar\left(p^+\right).$$

Replacing p^+ by its expression (see (3.3)), we get:

$$\begin{aligned} Cov\left(m^{mvs}(\eta, \overline{m}),\ p^+\right) &= Em^{mvs}(\eta, \overline{m})\, p^+ - Em^{mvs}(\eta, \overline{m})\, Ep^+ \\ &= l^{'}\Gamma^{-1}l - \overline{m}Ep^+. \end{aligned}$$

Therefore,

$$a = \frac{l^{'}\Gamma^{-1}l - \overline{m}Ep^+}{Var\left(p^+\right)}.$$

Second,

$$Cov\left(m^{mvs}(\eta, \overline{m}),\ p^{++}\right) = aCov\left(F_1, p^{++}\right) + bCov\left(F_2, p^{++}\right).$$

Replacing p^{++} by its expression (see (3.4)), we get:

$$Cov\left(m^{mvs}(\eta, \overline{m}),\ p^{++}\right) = Em^{mvs}(\eta, \overline{m})\, p^{++} - Em^{mvs}(\eta, \overline{m})\, Ep^{++} = \nu^{'}\Gamma^{-1}l - \overline{m}Ep^{++}.$$

Consequently,

$$aCov\left(F_1, p^{++}\right) + bCov\left(F_2, p^{++}\right) = \nu^{'}\Gamma^{-1}l - \overline{m}Ep^{++},$$

which implies that

$$b = \frac{\left(\nu^{'}\Gamma^{-1}l - \overline{m}Ep^{++}\right) - aCov\left(F_1, p^{++}\right)}{Cov\left(F_2, p^{++}\right)}.$$

Third,

$$Cov\left(m^{mvs}(\eta, \overline{m}),\ p^*\right) = aCov\left(F_1, p^*\right) + bCov\left(F_2, p^*\right) + cCov\left(F_3, p^*\right).$$

Replacing p^* by its expression (see (3.5)), we get:

$$Cov\left(m^{mvs}(\eta, \overline{m}),\ p^*\right) = Em^{mvs}(\eta, \overline{m})\, p^* - \overline{m}Ep^* = \eta^{'}\left[\Gamma^{(2)}\right]^{-1}\eta - \overline{m}Ep^*.$$

Consequently,

$$c = \frac{\left(\eta^{'}\left[\Gamma^{(2)}\right]^{-1}\eta - \overline{m}Ep^{*}\right) - aCov\left(F_1, p^{*}\right) - bCov\left(F_2, p^{*}\right)}{Cov\left(F_3, p^{*}\right)}.$$

It is obvious that the HJ stochastic discount factor can be written as:

$$m_{HJ} = \overline{m} + aF_1 + bF_2.$$

■

Proof of proposition 3.5. The linear regression of r_p^2 on r_p gives

$$r_p^2 = Er_p^2 + \frac{Cov\left(r_p, r_p^2\right)}{Var\left(r_p\right)}\left(r_p - Er_p\right) + \upsilon,$$

for any portfolio $r_p = \omega^{'}R$. Therefore, the cost of the squared portfolio return is

$$c^{*} = \overline{m}Er_p^2 + \frac{Cov\left(r_p, r_p^2\right)}{Var\left(r_p\right)}\left(1 - \overline{m}Er_p\right) + Cov\left(\upsilon, m^{mvs}\right).$$

If $Cov\left(\upsilon, m^{mvs}\right) = 0$, we have:

$$c^{*} = \overline{m}Er_p^2 + \frac{Cov\left(r_p, r_p^2\right)}{Var\left(r_p\right)}\left(1 - \overline{m}Er_p\right). \tag{A1}$$

Then, if $\omega_i = 1$ and $\omega_j = 0$ for $j \neq i$, equation (A1) implies that

$$\overline{\eta}_{ii} = \overline{m}ER_i^2 + \left(1 - \overline{m}ER_i\right)\frac{Cov\left(R_i, R_i^2\right)}{Var\left(R_i\right)} \text{ for } i\text{=1,...,}n.$$

For $\omega_i = \frac{1}{2}$, $\omega_j = \frac{1}{2}$, and $\omega_k = 0$ for $k \neq i$ and $k \neq j$. If we decompose the left-hand side of (A1), we have

$$\begin{aligned} c^{*} &= Em\left(\frac{1}{2}R_i + \frac{1}{2}R_j\right)^2 \\ &= \frac{1}{4}Em\left(R_i^2 + R_j^2 + 2R_iR_j\right) \\ &= \frac{1}{4}\left[EmR_i^2 + EmR_j^2 + 2EmR_iR_j\right] \\ &= \frac{1}{4}\overline{\eta}_{ii} + \frac{1}{4}\overline{\eta}_{jj} + \frac{1}{2}\overline{\eta}_{ij}, \end{aligned} \tag{A2}$$

where

$$\overline{\eta}_{ij} = EmR_iR_j.$$

We also decompose the right-hand side of (A1) and equate the resulting expression with (A2) to get

$$\begin{aligned} \overline{\eta}_{ij} = \; & \frac{1}{2}\overline{m}\left(ER_i^2 + ER_j^2 + 2ER_iR_j\right) + \\ & \frac{\left[1 - \frac{1}{2}\left(ER_i + ER_j\right)\overline{m}\right]Cov\left(\left(R_i + R_j\right), \left(R_i + R_j\right)^2\right)}{\left[Var\left(R_i\right) + Var\left(R_j\right) + 2Cov\left(R_i, R_j\right)\right]} - \\ & \frac{1}{2}\left(\overline{\eta}_{ii} + \overline{\eta}_{jj}\right) \end{aligned}$$

for $i \neq j$. But,

$$\begin{aligned} c^* &= Em^{mvs} r_p^2 \\ &= Em^{mvs}\left(\omega' R\right)^2 \\ &= Em^{mvs}\left(\omega^{(2)'} R^{(2)}\right) \\ &= \omega^{(2)'} Em^{mvs} R^{(2)} \\ &= \omega^{(2)'}\eta. \end{aligned}$$

Then,

$$c^* = \omega^{(2)'}\overline{\eta} = \overline{m} E r_p^2 + \frac{Cov\left(r_p, r_p^2\right)}{Var\left(r_p\right)}\left(1 - \overline{m} E r_p\right).$$

Conversely, let us assume that $\eta = \overline{\eta}$. We have,

$$\begin{aligned} c^* &= Em^{mvs} r_p^2 \\ &= Em^{mvs}\omega^{(2)'} R^{(2)} \\ &= \omega^{(2)'} Em^{mvs} R^{(2)} \\ &= \omega^{(2)'}\overline{\eta}. \end{aligned}$$

But we know that

$$\omega^{(2)'}\overline{\eta} = \overline{m} E r_p^2 + \frac{Cov\left(r_p, r_p^2\right)}{Var\left(r_p\right)}\left(1 - \overline{m} E r_p\right).$$

Therefore,

$$c^* = \overline{m} E r_p^2 + \frac{Cov\left(r_p, r_p^2\right)}{Var\left(r_p\right)}\left(1 - \overline{m} E r_p\right).$$

But we know that

$$c^* = \overline{m} E r_p^2 + \frac{Cov\left(r_p, r_p^2\right)}{Var\left(r_p\right)}\left(1 - \overline{m} E r_p\right) + Cov\left(\upsilon, m^{mvs}\right).$$

Consequently,

$$Cov\left(m^{mvs}, \upsilon\right) = c^* - \overline{m} E r_p^2 - \frac{Cov\left(r_p, r_p^2\right)}{Var\left(r_p\right)}\left(1 - \overline{m} E r_p\right) = 0.$$

■

Proof of Proposition 4.1. Assume that the joint process (m, R) is lognormal. This means that

$$\begin{bmatrix} Log\,(m) \\ Log\,(R) \end{bmatrix} \rightsquigarrow N\left[\begin{bmatrix} \mu_m \\ \mu_r \end{bmatrix}, \begin{bmatrix} \sigma_m^2 & \Sigma_{mr} \\ \Sigma_{mr} & \Sigma_r \end{bmatrix}\right].$$

We know that

$$EmR_i = 1 \ \forall i.$$

Let us compute

$$\eta_{ij} = EmR_iR_j.$$

Therefore,

$$Log\left(mR_iR_j\right) = Log\left(m\right) + Log\left(R_i\right) + Log\left(R_j\right).$$

Let μ_m and σ_m^2 denote the first two moments of $Log\left(m\right)$ and μ_i, and let σ_i^2 denote the first two moments of $Log\left(R_i\right)$. As a result,

$$\begin{aligned}
\eta_{ij} &= \exp\left[\mu_m + \mu_i + \mu_j + \frac{1}{2}\left(\sigma_i^2 + \sigma_j^2 + \sigma_m^2 + 2\sigma_{ij} + 2\sigma_{im} + 2\sigma_{jm}\right)\right] \\
&= \exp\left[\mu_m + \frac{1}{2}\sigma_m^2 + \mu_i + \frac{1}{2}\sigma_i^2 + \sigma_{im}\right]\exp\left[\mu_j + \frac{1}{2}\left(\sigma_j^2 + 2\sigma_{ij} + 2\sigma_{jm}\right)\right] \\
&= \left(EmR_i\right)\exp\left[\mu_j + \frac{1}{2}\left(\sigma_j^2 + 2\sigma_{ij} + 2\sigma_{jm}\right)\right] \\
&= \exp\left[\mu_j + \frac{1}{2}\left(\sigma_j^2 + 2\sigma_{ij} + 2\sigma_{jm}\right) + \mu_m + \frac{1}{2}\sigma_m^2\right]\exp\left[-\mu_m - \frac{1}{2}\sigma_m^2\right] \\
&= \exp\left[\mu_m + \frac{1}{2}\sigma_m^2 + \mu_j + \frac{1}{2}\sigma_j^2 + \sigma_{jm}\right]\exp\left[\frac{1}{2}\left(2\sigma_{ij}\right)\right]\exp\left[-\mu_m - \frac{1}{2}\sigma_m^2\right] \\
&= E\left(mR_j\right)\exp\left[\frac{1}{2}\left(2\sigma_{ij}\right)\right]\exp\left[-\mu_m - \frac{1}{2}\sigma_m^2\right].
\end{aligned}$$

But $E\left(mR_j\right) = 1$. Consequently,

$$\begin{aligned}
\eta_{ij} &= \exp\left[\frac{1}{2}\left(2\sigma_{ij}\right)\right]\exp\left[-\mu_m - \frac{1}{2}\sigma_m^2\right] \\
&= \frac{1}{\overline{m}}\exp\left[\frac{1}{2}\left(2\sigma_{ij}\right)\right] \\
&= \frac{1}{\overline{m}}\exp\left[\mu_i + \mu_j + \frac{1}{2}\left(\sigma_i^2 + \sigma_j^2 + 2\sigma_{ij}\right)\right]\exp\left[-\mu_i - \frac{1}{2}\sigma_i^2\right]\exp\left[-\mu_j - \frac{1}{2}\sigma_j^2\right] \\
&= \left[\frac{1}{\overline{m}}E\left(R_iR_j\right)\right]\left[\frac{1}{ER_iER_j}\right] \\
&= \frac{1}{\overline{m}}\frac{E\left(R_iR_j\right)}{ER_iER_j}.
\end{aligned}$$

■

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